Math 68. Algebraic Combinatorics.

Problem Set 4. Due on Friday, 11/8/2013.

1. How many SYT of shape (n^n) have main diagonal $(1, 4, 9, 16, \ldots, n^2)$?

2. Let $c(\lambda)$ denote the number of corner squares (or distinct parts) of the partition λ. For instance, $c(5, 5, 4, 2, 2, 2, 1, 1) = 4$. Show that

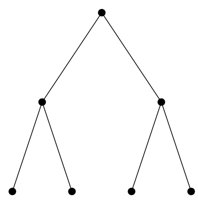
$$\sum_{\lambda \vdash n} c(\lambda) = p(0) + p(1) + \cdots + p(n-1),$$

   where $p(i)$ denotes the number of partitions of i (with $p(0) = 1$).

3. A $(0, 1)$-necklace of length n and weight i is a circular arrangement of i 1's and $n - i$ 0's. For instance, the $(0, 1)$-necklaces of length 6 and weight 3 are (writing a circular arrangement linearly) 000111, 001011, 010011, and 010101. (Cyclic shifts of a linear word represent the same necklace, e.g., 000111 is the same as 110001.) Let N_n denote the set of all $(0, 1)$-necklaces of length n. Define a partial order on N_n by letting $u \leq v$ if we can obtain v from u by changing some 0's to 1's. It's easy to see (you may assume it) that N_n is graded of rank n, with the rank of a necklace being its weight. Show that N_n is rank-symmetric, rank-unimodal, and Sperner.
   Hint: Show that $N_n \cong B_n/G$ for a suitable group G.

4. How many necklaces (up to cyclic symmetry) have n read beads and n blue beads? Express your answer as a sum over all divisors d of n.

5. Let Γ be the graph shown below.



   An automorphism of Γ is a permutation π of the vertices of Γ that preserves adjacencies (i.e., there is an edge between two vertices x and y if and only if there is an edge between $\pi(x)$ and $\pi(y)$). Let G be the automorphism group of Γ, so G has order 8.

   (a) What is the cycle index polynomial of G, acting on the vertices of Γ?

   (b) In how many ways can one color the vertices of Γ in n colors, up to symmetry of Γ?

6. For any finite group G of permutations of an ℓ-element set X, let $f(n)$ be the number of inequivalent (under the action of G) colorings of X with n colors. Find $\lim_{n\to\infty} f(n)/n^{\ell}$. Interpret your answer as saying that "most" colorings of X are asymmetric (have no symmetries).

7. Consider the group G of (orientation-preserving) symmetries of the cube.

   (a) Show that $|G| = 24$.

   (b) Find the number of inequivalent colorings of the faces of the cube using n colors.

   (c) Find the number of inequivalent colorings of the vertices of the cube using n colors.