Math 68. Algebraic Combinatorics.

Problem Set 3. Due on Friday, 10/25/2013.

1. Prove that
$$\frac{1}{1-z} = \prod_{j\geq 0}(1+z^{2^j}).$$

2. For fixed k, give the exponential generating function for the number of surjective maps from $[n]$ onto $[k]$.

3. (a) Let b_n denote the number of (labeled) rooted trees on the vertex set $[n]$ whose leaves (i.e., vertices with no children) are colored either red of blue. Find an equation satisfied by the exponential generating function
$$B(z) = \sum_{n\geq 0} b_n \frac{z^n}{n!} = 2z + 4\frac{z^2}{2!} + 24\frac{z^3}{3!} + \ldots$$

   (b) Use the Lagrange inversion formula to deduce that
$$b_n = \sum_{k=0}^{n} \binom{n}{k} k^{n-1}.$$

   (c) * Give a direct combinatorial proof of (b).

4. Describe an explicit bijection between the set of (unlabeled) rooted plane trees with n edges and the set of Dyck paths with $2n$ steps.

5. Let $M(n)$ be the set of all subsets of $[n]$, with the ordering $A \leq B$ if the elements of A are $a_1 > a_2 > \cdots > a_j$ and the elements of B are $b_1 > b_2 > \cdots > b_k$, where $j \leq k$ and $a_i \leq b_i$ for $1 \leq i \leq j$. (The empty set $\emptyset$ is the bottom element of $M(n)$.)

   (a) Draw the Hasse diagrams (with vertices labeled by the subsets they represent) of $M(1)$, $M(2)$, $M(3)$, and $M(4)$.

   (b) Show that $M(n)$ is graded of rank $\binom{n+1}{2}$. What is $\text{rank}(\{a_1, \ldots, a_k\})$?

   (c) Define the *rank-generating function* of a graded poset P to be
$$F(P, q) := \sum_{x\in P} q^{\text{rank}(x)}.$$

   Show that the rank-generating function of $M(n)$ is given by
$$F(M(n), q) = (1+q)(1+q^2)\cdots(1+q^n).$$

6. * Let h_n be the number of ways to choose a permutation π of $[n]$ and a subset S of $[n]$ such that if $i \in S$, then $\pi(i) \notin S$. Find an expression for the exponential generating function $\sum_{n\geq 0} h_n \frac{z^n}{n!}$.

7. ** (Extra credit) If you solved problem 7 in Problem Set 2 bijectively, as part of your solution you probably found a bijection between paths with $2n$ steps $N = (0,1)$ and $E = (1,0)$ starting at the origin and ending at (n,n) (sometimes called Grand Dyck paths, and counted by $\binom{2n}{n}$) and paths with $2n$ steps N and E starting at the origin and not going below $y = x$ (sometimes called ballot paths). Concatenating a ballot path with its reflection gives a trivial bijection between ballot paths of length $2n$ and symmetric Dyck paths of length $4n$.

   Define the *height* of an occurrence of NNE in a lattice path to be one plus the number of N steps before the occurrence. For example, $NNEENNEE$ has occurrences of NNE at heights 1 and 3. Define the *position* of an occurrence of NE to be one plus the total number of steps before the occurrence. For example, $NENE$ has occurrences of NE in positions 1 and 3.

   Find a bijection between symmetric Dyck paths of length $4n$ and Grand Dyck paths of length $2n$ that maps the heights of the occurrences of NNE to the positions of the occurrences of NE.

   (As an example, note that the heights of the occurrences of NNE in the 6 symmetric Dyck paths of length 8 are $\emptyset, \{1\}, \{2\}, \{2\}, \{3\}, \{1,3\}$, which are also the positions of the occurrences of NE in Grand Dyck paths of length 4 .)