Math 68. Algebraic Combinatorics.

Problem Set 2. Due on Friday, 10/11/2013.

1. Find the ordinary generating function of the sequence $a_n = 2 \cdot 3^n - n^2$ (for $n \geq 0$) in a simple, closed form.

2. Consider the recurrence $a_{n+3} = 3a_{n+2} - 4a_n$, with initial conditions $a_0 = 1$, $a_1 = 2$, $a_2 = 6$. Find the ordinary generating function $\sum_{n \geq 0} a_n z^n$ and the expression of the general term a_n.

3. Find a generating function $A(z)$ such that the coefficient of z^{100} is the number of ways to give change of a dollar using cents, nickels, dimes, and quarters.

4. Prove that the number of compositions of n with an even number of parts is 2^{n-2}.

5. Given two sequences $\{a_n\}_{n \geq 0}$ and $\{b_n\}_{n \geq 0}$, its Hadamard product is the sequence $\{a_n b_n\}_{n \geq 0}$. Show that if $\{a_n\}_{n \geq 0}$ and $\{b_n\}_{n \geq 0}$ have rational generating functions, then so does their Hadamard product.

6. Find an expression for $S(n, k)$ (the Stirling number of the second kind) by extracting the coefficient of z^n in the exponential generating function for set partitions with k blocks.

7. Prove that
$$\sum_{k=0}^{n} \binom{2k}{k} \binom{2(n-k)}{n-k} = 4^n.$$

   Hint: There is a bijective proof, but if you can't find it, try using generating functions instead.