Combinations

April 12, 2006

Binomial Coefficients

Definition. *The number of distinct subsets with* j *elements that can be chosen from a set with* n *elements is denoted by* $\binom{n}{j}$. *The number* $\binom{n}{j}$ *is called a binomial* coefficient.

Recurrence Relation

Theorem. *For integers n and j, with $0 < j < n$, the binomial coefficients satisfy:*

$$\binom{n}{j} = \binom{n-1}{j} + \binom{n-1}{j-1} .$$

Pascal's triangle

	j = 0	1	2	3	4	5	6	7	8	9	10
n = 0	1										
1	1	1									
2	1	2	1								
3	1	3	3	1							
4	1	4	6	4	1						
5	1	5	10	10	5	1					
6	1	6	15	20	15	6	1				
7	1	7	21	35	35	21	7	1			
8	1	8	28	56	70	56	28	8	1		
9	1	9	36	84	126	126	84	36	9	1	
10	1	10	45	120	210	252	210	120	45	10	1

Theorem. *The binomial coefficients are given by the formula*

$$\binom{n}{j} = \frac{(n)_j}{j!} = \frac{n!}{j!(n-j)!}.$$

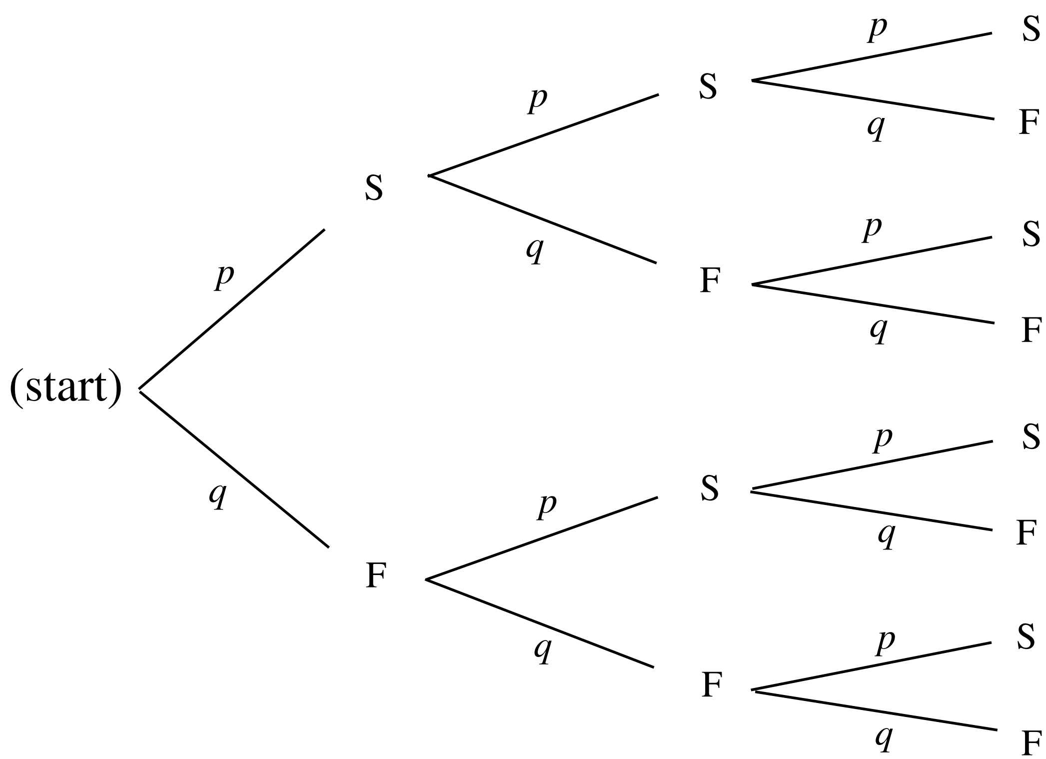
Bernoulli Trials

Definition. *A* Bernoulli trials process *is a sequence of n chance experiments such that*

1. Each experiment has two possible outcomes, which we may call *success* and *failure*.

2. The probability p of success on each experiment is the same for each experiment, and this probability is not affected by any knowledge of previous outcomes. The probability q of failure is given by $q = 1 - p$.

Tree diagram



ω	$m(\omega)$
ω_1	p^3
ω_2	p^2q
ω_3	p^2q
ω_4	$p\,q^2$
ω_5	$q\,p^2$
ω_6	q^2p
ω_7	q^2p
ω_8	q^3

Binomial Probabilities

We denote by $b(n, p, j)$ the probability that in n Bernoulli trials there are exactly j successes.

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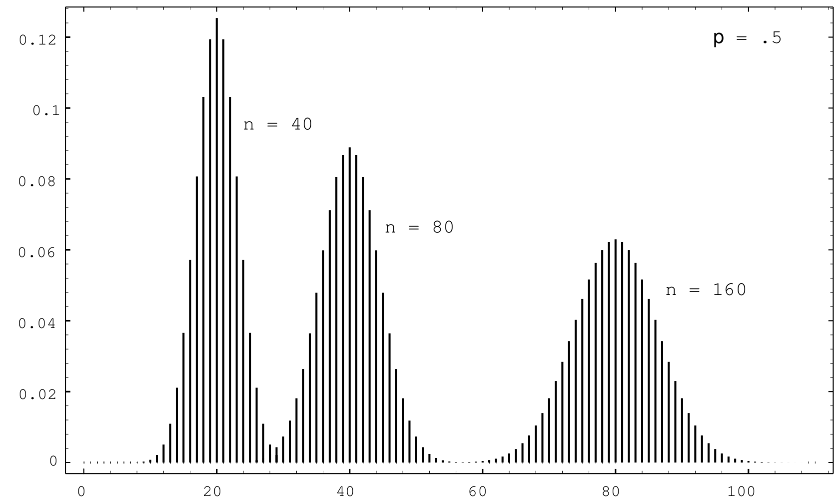
Theorem. *Given n Bernoulli trials with probability p of success on each experiment, the probability of exactly j successes is*

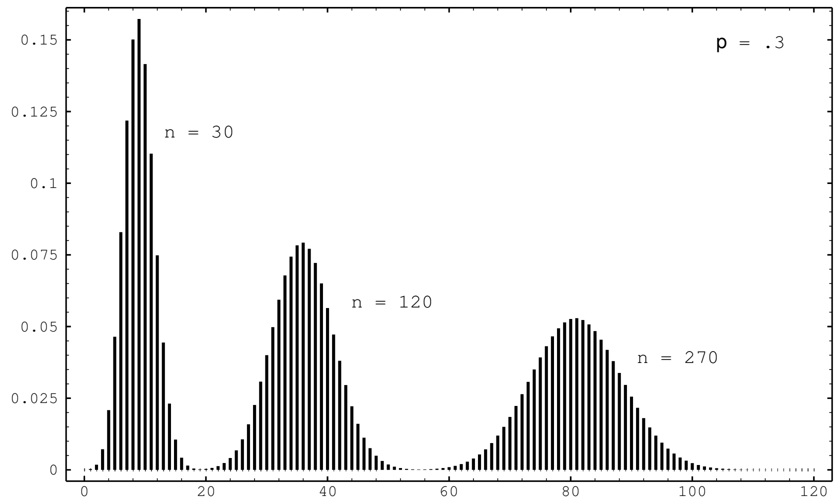
$$b(n, p, j) = \binom{n}{j} p^j q^{n-j}$$

where $q = 1 - p$.

Binomial Distributions

Definition. *Let n be a positive integer, and let p be a real number between 0 and 1. Let B be the random variable which counts the number of successes in a Bernoulli trials process with parameters n and p. Then the distribution $b(n, p, j)$ of B is called the* binomial distribution.


p = .5
n = 40
n = 80
n = 160
0.12
0.1
0.08
0.06
0.04
0.02
0
0
20
40
60
80
100


p = .3
n = 30
n = 120
n = 270
0.15
0.125
0.1
0.075
0.05
0.025
0
0
20
40
60
80
100
120

Binomial Expansion

Theorem. *The quantity* $(a+b)^n$ *can be expressed in the form*

$$(a+b)^n = \sum_{j=0}^{n} \binom{n}{j} a^j b^{n-j} \ .$$

Corollary. *The sum of the elements in the nth row of Pascal's triangle is 2^n. If the elements in the nth row of Pascal's triangle are added with alternating signs, the sum is 0.*

Inclusion-Exclusion Principle

Theorem. *Let P be a probability distribution on a sample space Ω, and let $\{A_1,\ A_2,\ \ldots,\ A_n\}$ be a finite set of events. Then*

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) = \sum_{i=1}^{n} P(A_i) \; - \sum_{1 \le i < j \le n} P(A_i \cap A_j)$$

$$+ \sum_{1 \le i < j < k \le n} P(A_i \cap A_j \cap A_k) - \cdots \; .$$

That is, to find the probability that at least one of n events A_i occurs, first add the probability of each event, then subtract the probabilities of all possible two-way intersections, add the probability of all three-way intersections, and so forth.

Hat Check Problem

In a restaurant n hats are checked and they are hopelessly scrambled; what is the probability that no one gets his own hat back?

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- If A_i is the event that the ith element a_i remains fixed under this map, then

$$P(A_i) = \frac{1}{n}.$$

- If we fix a particular pair (a_i, a_j), then

$$P(A_i \bigcap A_j) = \frac{1}{n(n-1)}.$$

- The number of terms of the form $P(A_i \bigcap A_j)$ is $\left(\begin{array}{c} n \\ 2 \end{array} \right)$.

- ‘For any three events A_1, A_2, A_3

$$P(A_i \cap A_j \cap A_k) = \frac{(n-3)!}{n!} = \frac{1}{n(n-1)(n-2)} \ ,$$

and the number of such terms is

$$\binom{n}{3} = \frac{n(n-1)(n-2)}{3!} \ .$$

- Hence

$$P(\text{at least one fixed point}) = 1 - \frac{1}{2!} + \frac{1}{3!} - \cdots (-1)^{n-1}\frac{1}{n!}$$

and

$$P(\text{no fixed point}) = \frac{1}{2!} - \frac{1}{3!} + \cdots (-1)^n \frac{1}{n!} \ .$$

n	Probability that no one gets his own hat back
3	.333333
4	.375
5	.366667
6	.368056
7	.367857
8	.367882
9	.367879
10	.367879

Problems

Show that the number of ways that one can put n different objects into three boxes with a in the first, b in the second, and c in the third is $n!/(a!\,b!\,c!)$.

Suppose that a die is rolled 20 independent times, and each time we record whether or not the event $\{2, 3, 5, 6\}$ has occurred.

1. What is the distribution of the number of times this event occurs in 20 rolls?

2. Calculate the probability that the event occurs five times.

Suppose that a basketball player sinks a basket from a certain position on the court with probability 0.35.

1. What is the probability that the player sinks three baskets in ten independent throws?

2. What is the probability that the player throws ten times before obtaining the first basket?

3. What is the probability that the player throws ten times before obtaining two baskets?