Infinite Sample Spaces

04/03/2006

Odds

- **March Madness:** Tonight UCLA vs Florida.
- What are the odds that UCLA wins?
- Then, what is the probability that UCLA wins?

- If $P(E) = p$, the *odds* in favor of the event E occurring are $r : s$ (r to s) where $r/s = p/(1-p)$.

- If r and s are given, then p can be found by using the equation $p = r/(r+s)$.

Infinite Sample Space

- If
$$\Omega = \{\omega_1, \omega_2, \omega_3, \ldots\}$$
is a countably infinite sample space, then a distribution function is a function which satisfies

  1. $m(\omega) \geq 0$, for all $\omega \in \Omega$, and
  2. $\sum_{\omega \in \Omega} m(\omega) = 1$.

- The sum in the second part must be *convergent*.

Examples

A coin is tossed until the first time that a head turns up. Let the outcome of the experiment, ω, be the first time that a head turns up (we assume that, eventually, it would come up).

- Describe the sample space and the distribution function.
- What is the probability that the coin turns up heads in an even number of tosses?
- What is the probability that the coin turns up heads on the fifth toss?
- What is the probability that the coin turns up heads after the fifth toss?

Continuous Probability Densities

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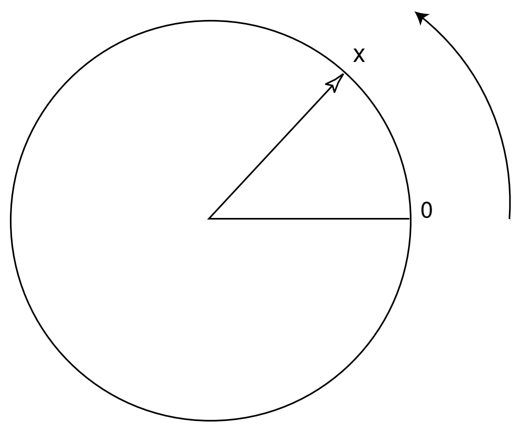
- Consider the experiment of tossing a fair coin repeatedly.
- Allow the possibility of never terminating the experiment.
- What would be an appropriate sample space for the experiment?

Example: The Spinner

- Let us construct a spinner, which consists of a circle of unit circumference and a pointer.



- The experiment consists of spinning the pointer and recording the label of the point at the tip of the pointer.

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- The probability

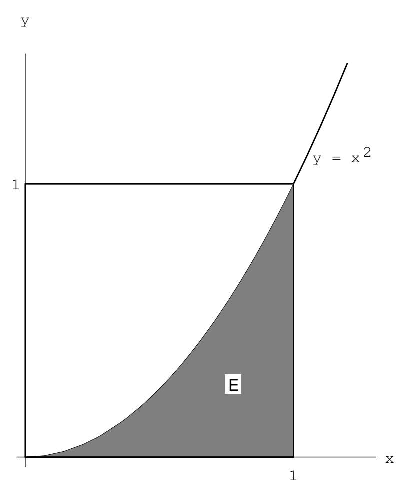
$$P(0 \leq X \leq 1)$$

should be equal to 1.

Monte Carlo Procedure and Areas

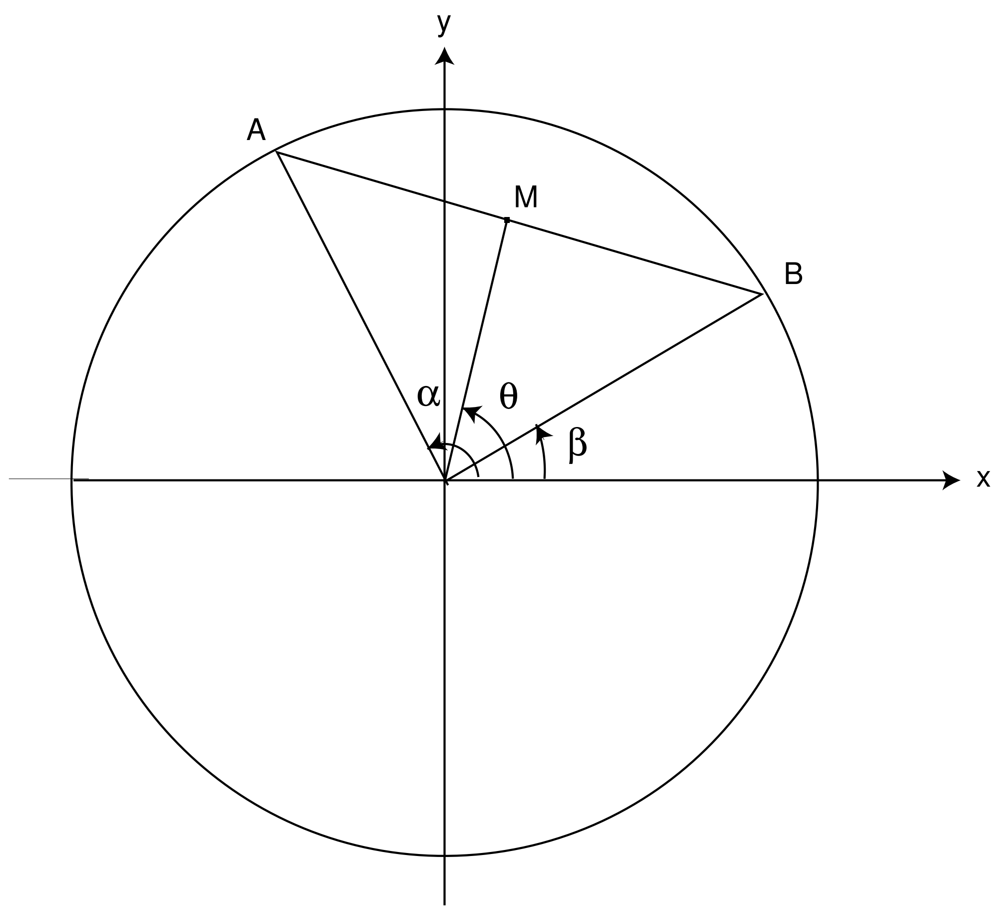
- A *Monte Carlo procedure* is a procedure which involves chance and it is used to estimate quantities whose exact values are difficult or impossible to calculate exactly.

- For example it can be used to estimate areas of plane figures.



Bertrand's Paradox

- A chord of a circle is a line segment both of whose endpoints lie on the circle.
- Suppose that a chord is drawn *at random* in a unit circle.
- What is the probability that its length exceeds $\sqrt{3}$?


y
A
M
B
α
θ
β
x

We can describe each chord by giving:

1. The rectangular coordinates (x, y) of the midpoint M;

2. The polar coordinates (r, θ) of the midpoint M;

3. The polar coordinates $(1, \alpha)$ and $(1, \beta)$ of the endpoints A and B;

1. To simulate this case, we choose values for x and y from $[-1, 1]$ at random. Then we check whether $x^2 + y^2 \leq 1$. If not, the point $M = (x, y)$ lies outside the circle and cannot be the midpoint of any chord, and we ignore it. Otherwise, M lies inside the circle and is the midpoint of a unique chord, whose length L is given by the formula:
$$L = 2\sqrt{1 - (x^2 + y^2)}\ .$$

2. To simulate this case, we take account of the fact that any rotation of the circle does not change the length of the chord, so we might as well assume in advance that the chord is horizontal. Then we choose r from $[-1, 1]$ at random, and compute the length of the resulting chord with midpoint $(r, \pi/2)$ by the formula:

$$L = 2\sqrt{1 - r^2}\ .$$

3. To simulate this case, we assume that one endpoint, say B, lies at $(1, 0)$ (i.e., that $\beta = 0$). Then we choose a value for α from $[0, 2\pi]$ at random and compute the length of the resulting chord, using the Law of Cosines, by the formula:

$$L = \sqrt{2 - 2\cos\alpha}\ .$$

1. $L > \sqrt{3}$ if(x, y) lies inside a circle of radius 1/2, which occurs with probability
$$p = \frac{\pi(1/2)^2}{\pi(1)^2} = \frac{1}{4}\ .$$

2. $L > \sqrt{3}$ if $|r| < 1/2$, which occurs with probability
$$\frac{1/2 - (-1/2)}{1 - (-1)} = \frac{1}{2}\ .$$

3. $L > \sqrt{3}$ if $2\pi/3 < \alpha < 4\pi/3$, which occurs with probability
$$\frac{4\pi/3 - 2\pi/3}{2\pi - 0} = \frac{1}{3}\ .$$