Gambler's Ruin

May 22, 2006

- Assume that p and q are non-negative real numbers with $p+q = 1$.

- The common distribution function of the jumps of the random walk is

$$f_X(x) = \begin{cases} p, & \text{if } x = 1, \\ q, & \text{if } x = -1. \end{cases}$$

The Gambler's Ruin Problem

- A gambler starts with a "stake" of size s.
- She plays until her capital reaches the value M or the value 0.
- What is the probability of occurence of each of these two outcomes?

- Defining q_k to be the probability that the gambler's stake reaches 0 before it reaches M, given that the initial stake is k.

$$q_k = pq_{k+1} + qq_{k-1} \ .$$

- Therefore, if $p \neq q$,

$$q_z = 1 - \frac{(q/p)^z - 1}{(q/p)^M - 1} = \frac{(q/p)^M - (q/p)^z}{(q/p)^M - 1} \; .$$

- If $p = q = 1/2$,

$$q_z = \frac{M - z}{M} \; .$$

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- Thus, for all z, it is the case that $p_z + q_z = 1$.

Infinitely Rich Adversaries

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- If $M \to \infty$, we see that the probability of eventual ruin tends to 1.

Paradoxes

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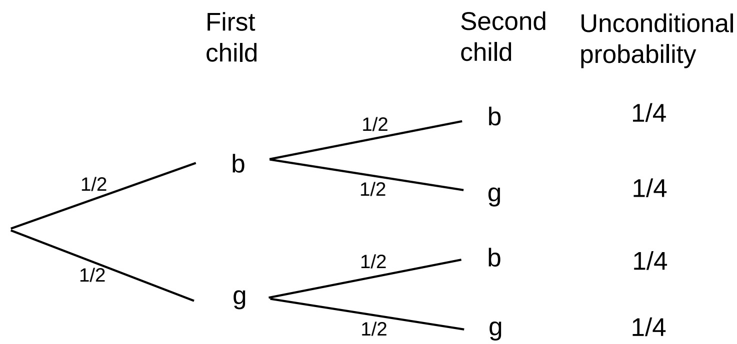
First child
Second child
Unconditional probability
1/2
b
1/2
1/2
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1/4
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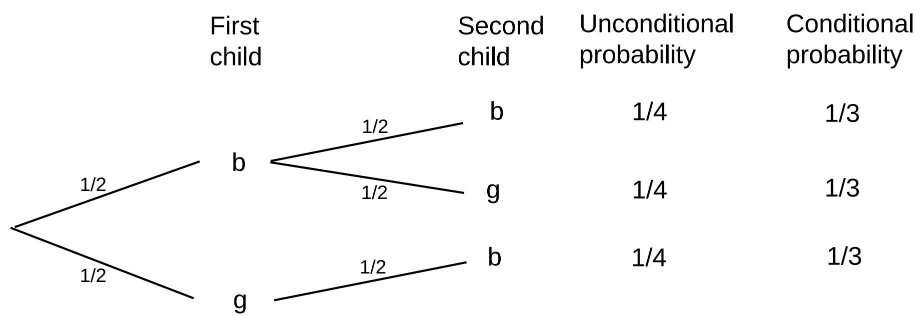
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- Mr. Smith is the father of two. We meet him walking along the street with a young boy whom he proudly introduces as his son. What is the probability that Mr. Smith's other child is also a boy?

- A shopkeeper says she has two new baby beagles to show you, but she doesn't know whether they're both male, both female, or one of each sex.

- You tell her that you want only a male, and she telephones the fellow who's giving them a bath.

- "Is at least one a male?" she asks.

- "Yes," she informs you with a smile.

- What is the probability that the *other* one is male?

- There are two envelopes on the table. One contains some money, and the other contains twice as much money, but you don't know which.

  1. You pick up one envelope, open it, and see that it contains \$10. What is the probability that the other envelope contains \$5? What is the probability that it has \$20?

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  1. You pick up one envelope, open it, and see that it contains \$10. What is the probability that the other envelope contains \$5? What is the probability that it has \$20?
  2. Using your answer to (1), what is the expected (average) value of the other envelope?
  3. From your answer to (2), would you prefer to keep the \$10 or switch envelopes, or are they both equally good?

4. Now think about the problem from another person's point of view. They picked up the other envelope and saw \$ $x$in it. So they think your envelope has either $\$2x$ or $\$x/2$ in it. What do they think is the expected value of your envelope? Would they prefer to keep the $\$x$ or switch envelopes?

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5. Does this make sense? Can you explain it?

- You and your sibling [brother or sister] both get great presents on your birthdays. You argue with each other about who got the most expensive present. You decide that to find out, you will go to the store and see how much each present cost. Whoever?s present is the most expensive wins the argument. To make up for that, the winner of the argument will give their present to the loser.

- Thus, if you win the argument, you have to give up your present. But if you lose the argument, you get your sibling?s present, which is more expensive than yours. So the expected value is in your favor.

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- On the other hand, your sibling can use exactly the same logic! Try to resolve this contradiction.