Random Walks

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Random Walks in Euclidean Space

Definition. *Let* $\{X_k\}_{k=1}^{\infty}$ *be a sequence of independent, identically distributed discrete random variables. For each positive integer* n*, we let* S_n *denote the sum* $X_1 + X_2 + \cdots + X_n$*. The sequence* $\{S_n\}_{n=1}^{\infty}$ *is called a* random walk. *If the common range of the* X_k*'s is* $\mathbf{R}^m$*, then we say that* $\{S_n\}$ *is a random walk in* $\mathbf{R}^m$*.*

Example

- One can imagine that a particle is placed at the origin in $\mathbf{R}^m$ at time $n = 0$.
- The sum S_n represents the position of the particle at the end of n seconds.
- Thus, in the time interval $[n-1, n]$, the particle moves (or jumps) from position S_{n-1} to S_n.
- The vector representing this motion is just $S_n - S_{n-1}$, which equals X_n.

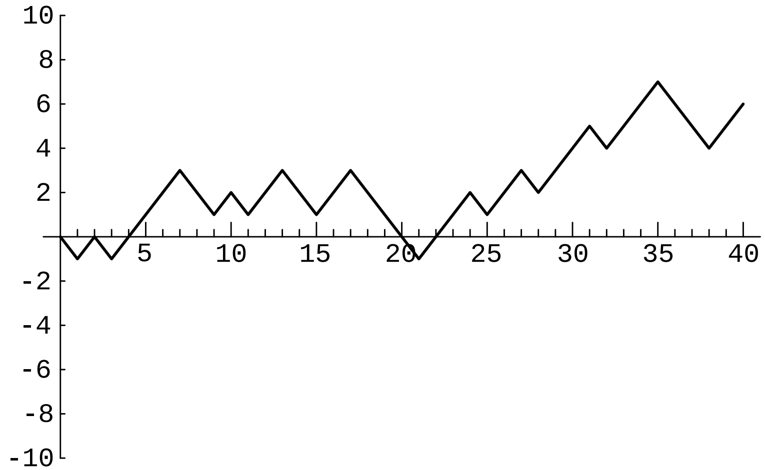
- Another model of a random walk is a game, involving two people, which consists of a sequence of independent, identically distributed moves.

- The sum S_n represents the score of the first person, say, after n moves, with the assumption that the score of the second person is $-S_n$.

Random Walks on the Real Line

- The common distribution function of the random variables X_n is given by

$$f_X(x) = \begin{cases} 1/2, & \text{if } x = \pm 1, \\ 0, & \text{otherwise.} \end{cases}$$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Returns and First Returns

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Theorem. *The probability of a return to the origin at time* $2m$ *is given by*

$$u_{2m} = \binom{2m}{m} 2^{-2m} \ .$$

The probability of a return to the origin at an odd time is 0.

- A random walk is said to have a *first return* to the origin at time $2m$ if $m > 0$, and $S_{2k} \neq 0$ for all $k < m$.

- We define f_{2m} to be the probability of this event.

Theorem. *For* $n \geq 1$*, the probabilities* $\{u_{2k}\}$ *and* $\{f_{2k}\}$ *are related by the equation*

$$u_{2n} = f_0 u_{2n} + f_2 u_{2n-2} + \cdots + f_{2n} u_0 \ .$$

Theorem. *For $m \geq 1$, the probability of a first return to the origin at time $2m$ is given by*

$$f_{2m} = \frac{u_{2m}}{2m-1} = \frac{\binom{2m}{m}}{(2m-1)2^{2m}} \ .$$

Proof. Define the generating functions

$$U(x) = \sum_{m=0}^{\infty} u_{2m} x^m$$

and

$$F(x) = \sum_{m=0}^{\infty} f_{2m} x^m \ .$$

$\square$

Probability of Eventual Return

- In the symmetric random walk process in $\mathbf{R}^m$, what is the probability that the particle eventually returns to the origin?

Eventual Return in $\mathbf{R}^1$

- We will define w_n to be the probability that a first return has occurred no later than time n.

- Define the probability that the particle eventually returns to the origin to be
$$w_* = \lim_{n \to \infty} w_n \ .$$

- In terms of the f_n probabilities, we see that

$$w_{2n} = \sum_{i=1}^{n} f_{2i} \ .$$

Theorem. *With probability one, the particle returns to the origin.*

Eventual Return in $\mathbf{R}^m$

- We define $f_{2n}^{(m)}$ to be the probability that the first return to the origin in $\mathbf{R}^m$ occurs at time $2n$.

- The quantity $u_{2n}^{(m)}$ is defined in a similar manner.

- For all $m \geq 1$,

$$u_{2n}^{(m)} = f_0^{(m)} u_{2n}^{(m)} + f_2^{(m)} u_{2n-2}^{(m)} + \cdots + f_{2n}^{(m)} u_0^{(m)} \ .$$

- Define

$$U^{(m)}(x) = \sum_{n=0}^{\infty} u_{2n}^{(m)} x^n$$

and

$$F^{(m)}(x) = \sum_{n=0}^{\infty} f_{2n}^{(m)} x^n \ .$$

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$$w_*^{(m)} = \lim_{x \uparrow 1} F^{(m)}(x) = \lim_{x \uparrow 1} \frac{U^{(m)}(x) - 1}{U^{(m)}(x)} \ ,$$

- In $\mathbb{R}^2$ the probability of eventual return is 1.
- In $\mathbb{R}^3$ the probability of eventual return is *strictly* less than 1.