Expectation and Variance: Continuous Random Variables

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Expected Value

Definition. *Let X be a real-valued random variable with density function $f(x)$. The* expected value $\mu = E(X)$ *is defined by*

$$\mu = E(X) = \int_{-\infty}^{+\infty} x f(x)\, dx \ ,$$

provided the integral

$$\int_{-\infty}^{+\infty} |x| f(x)\, dx$$

is finite.

Properties

- If X and Y are real-valued random variables and c is any constant, then

$$
\begin{aligned}
E(X+Y) &= E(X)+E(Y) \, , \\
E(cX) &= cE(X) \, .
\end{aligned}
$$

- More generally, if X_1, X_2, . . . , X_n are n real-valued random variables, and c_1, c_2, . . . , c_n are n constants, then

$$E(c_1X_1+c_2X_2+\cdots+c_nX_n) = c_1E(X_1)+c_2E(X_2)+\cdots+c_nE(X_n) \, .$$

Example

- Suppose Mr. and Mrs. Lockhorn agree to meet at the Hanover Inn between 5:00 and 6:00 P.M. on Tuesday.

- Suppose each arrives at a time between 5:00 and 6:00 chosen at random with uniform probability.

- Let Z be the random variable which describes the length of time that the first to arrive has to wait for the other.

- What is $E(Z)$?

Expectation of a Function of a Random Variable

Theorem. *If X is a real-valued random variable and if $\phi : \boldsymbol{R} \to \boldsymbol{R}$ is a continuous real-valued function, then*

$$E(\phi(X)) = \int_{-\infty}^{+\infty} \phi(x) f_X(x)\, dx \ ,$$

provided the integral exists.

Expectation of the Product of Two Random Variables

Theorem. *Let X and Y be independent real-valued continuous random variables with finite expected values. Then we have*

$$E(XY) = E(X)E(Y) \; .$$

Example

- Let $Z = (X, Y)$ be a point chosen at random in the unit square.
- What is $E(X^2Y^2)$?

Variance

Definition. *Let* X *be a real-valued random variable with density function* $f(x)$*. The* variance $\sigma^2 = V(X)$ *is defined by*

$$\sigma^2 = V(X) = E((X - \mu)^2)\ .$$

Computation

Theorem. *If X is a real-valued random variable with $E(X) = \mu$, then*

$$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx \ .$$

Properties of the variance

- If X is a real-valued random variable defined on Ω and c is any constant, then

$$\begin{aligned} V(cX) &= c^2 V(X) \ , \\ V(X + c) &= V(X) \ . \end{aligned}$$

- If X is a real-valued random variable with $E(X) = \mu$, then

$$V(X) = E(X^2) - \mu^2 \ .$$

- If X and Y are independent real-valued random variables on Ω, then
$$V(X+Y) = V(X) + V(Y) \ .$$

Example

- Let X be an exponentially distributed random variable with parameter λ.

- Then the density function of X is

$$f_X(x) = \lambda e^{-\lambda x} \ .$$

- What is $E(X)$ and $V(X)$?

Normal Density

- Let Z be a standard normal random variable with density function

$$f_Z(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \; .$$

- What us $E(X)$ and $V(X)$?

Cauchy Density

- Let X be a continuous random variable with the Cauchy density function

$$f_X(x) = \frac{a}{\pi} \frac{1}{a^2 + x^2} .$$

- What is $E(X)$ and $V(X)$?

Independent Trials

Theorem. *If X_1, X_2, . . . , X_n is an independent trials process of real-valued random variables, with $E(X_i) = \mu$ and $V(X_i) = \sigma^2$, and if*

$$\begin{aligned} S_n &= X_1 + X_2 + \cdots + X_n \ , \\ A_n &= \frac{S_n}{n} \ , \end{aligned}$$

then

$$\begin{aligned} E(S_n) &= n\mu \ , \\ E(A_n) &= \mu \ , \\ V(S_n) &= n\sigma^2 \ , \end{aligned}$$

$$V(A_n) = \frac{\sigma^2}{n} .$$

It follows that if we set

$$S_n^* = \frac{S_n - n\mu}{\sqrt{n\sigma^2}} ,$$

then

$$E(S_n^*) = 0 ,$$
$$V(S_n^*) = 1 .$$

We say that S_n^* *is a* standardized version *of* S_n