# Important Densities

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# Continuous Uniform Density

- Let $U$ be the random variable whose value represents the outcome of the experiment consisting of choosing a real number at random from the interval $[a, b]$.

$$f(\omega) = \begin{cases} 1/(b-a), & \text{if } a \leq \omega \leq b, \\ 0, & \text{otherwise.} \end{cases}$$

# Exponential Density

- The exponential density function is defined by

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } 0 \leq x < \infty, \\ 0, & \text{otherwise.} \end{cases}$$

- $\lambda$ is any positive constant, depending on the experiment.

# The cumulative distribution function

- Let $T$ be an exponentially distributed random variable with parameter $\lambda$.
- If $x \geq 0$, then we have

$$\begin{aligned} F(x) &= P(T \leq x) \\ &= \int_0^x \lambda e^{-\lambda t}\, dt \\ &= 1 - e^{-\lambda x} \ . \end{aligned}$$

# The "Memoryless" Property

$$P(T > r + s \,|\, T > r) = P(T > s)\ .$$

# Gamma Density

- Define $X_1,\ X_2,\ \ldots$ to be a sequence of independent exponentially distributed random variables with parameter $\lambda$.
- Consider a time interval of length $t$.
- Let $Y$ denote the random variable which counts the number of emissions that occur in the time interval.

- Let $S_n$ denote the sum $X_1 + X_2 + \cdots + X_n$

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$$= P(S_n \leq t) - P(S_{n+1} \leq t) \ .$$

- The density of $S_n$ is called *the gamma density* with parameters $\lambda$ and $n$:

$$g_n(x) = \begin{cases} \lambda \frac{(\lambda x)^{n-1}}{(n-1)!} e^{-\lambda x}, & \text{if } x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

- The cumulative distribution function is

$$G_n(x) = \begin{cases} 1 - e^{-\lambda x}\left(1 + \frac{\lambda x}{1!} + \cdots + \frac{(\lambda x)^{n-1}}{(n-1)!}\right), & \text{if } x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

- Then

$$P(Y = n) = e^{-\lambda t} \frac{(\lambda t)^n}{n!} .$$

# Example

- Suppose that customers arrive at random times at a service station with one server, and suppose that each customer is served immediately if no one is ahead of him, but must wait his turn in line otherwise.

- How long should each customer expect to wait?

- Assume that the interarrival times between successive customers are given by random variables $X_1$, $X_2$, . . . , $X_n$ with an exponential cumulative distribution function given by

$$F_X(t) = 1 - e^{-\lambda t}.$$

- Assume, too, that the service times for successive customers are given by random variables $Y_1$, $Y_2$, . . . , $Y_n$

$$F_Y(t) = 1 - e^{-\mu t}.$$

# Functions of a Random Variable

**Theorem.** *Let* $X$ *be a continuous random variable, and suppose that* $\phi(x)$ *is a strictly increasing function on the range of* $X$*. Define* $Y = \phi(X)$*. Suppose that* $X$ *and* $Y$ *have cumulative distribution functions* $F_X$ *and* $F_Y$ *respectively. Then these functions are related by*

$$F_Y(y) = F_X(\phi^{-1}(y)).$$

*If* $\phi(x)$ *is strictly decreasing on the range of* $X$*, then*

$$F_Y(y) = 1 - F_X(\phi^{-1}(y)) \ .$$

**Corollary.** *Let $X$ be a continuous random variable, and suppose that $\phi(x)$ is a strictly increasing function on the range of $X$. Define $Y = \phi(X)$. Suppose that the density functions of $X$ and $Y$ are $f_X$ and $f_Y$, respectively. Then these functions are related by*

$$f_Y(y) = f_X(\phi^{-1}(y))\frac{d}{dy}\phi^{-1}(y) \ .$$

*If $\phi(x)$ is strictly decreasing on the range of $X$, then*

$$f_Y(y) = -f_X(\phi^{-1}(y))\frac{d}{dy}\phi^{-1}(y) \ .$$

# Simulation

**Corollary.** *If $F(y)$ is a given cumulative distribution function that is strictly increasing when $0 < F(y) < 1$ and if $U$ is a random variable with uniform distribution on $[0, 1]$, then*

$$Y = F^{-1}(U)$$

*has the cumulative distribution $F(y)$.*

# Normal Density

- The normal density function with parameters $\mu$ and $\sigma$ is defined as follows:

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2} \ .$$

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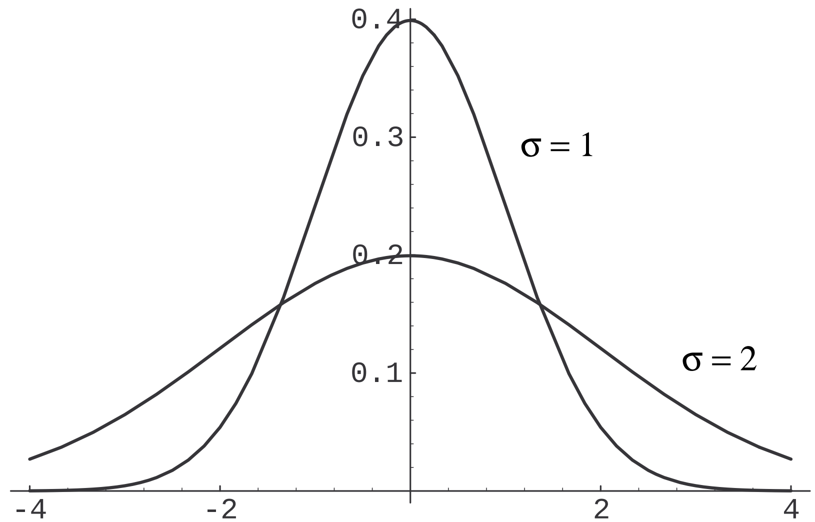
- The parameter $\mu$ represents the "center" of the density.

- The parameter $\sigma$ is a measure of the "spread" of the density.

# The Cumulative Distribution

- The cumulative distribution function is given by the formula

$$F_X(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}\sigma} e^{-(u-\mu)^2/2\sigma^2} \, du \ .$$


0.4
0.3
0.2
0.1
σ = 1
σ = 2
-4
-2
2
4

# The Standard Normal Random Variable

- A normal random variable with parameters $\mu = 0$ and $\sigma = 1$ is said to be a *standard* normal random variable.

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- If we write
$$X = \sigma Z + \mu ,$$
then $X$ is a normal random variable with parameters $\mu$ and $\sigma$.

- The cumulative distribution of $X$ in terms of $Z$ is
$$F_X(x) \quad = \quad F_Z\left(\frac{x - \mu}{\sigma}\right) \; .$$