Conditional Probability

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- Suppose that we draw two cards successively without replacement from a standard deck D.

- Consider the event $A = \{$the second card is a king$\}$. What is $P(A)$?

- Suppose that you are told after the first card is drawn that is was a king. What is the probability $P(A|B)$ that the second card is a king?

Definition

Let $\Omega = \{\omega_1, \omega_2, \ldots, \omega_r\}$ be the original sample space with distribution function $m(\omega_j)$ assigned. Suppose we learn that the event E has occurred.

- If a sample point ω_j is not in E, we want $m(\omega_j | E) = 0$.

- For ω_k in E, we should have the same relative magnitudes that they had before we learned that E had occurred:

$$m(\omega_k | E) = cm(\omega_k).$$

But we must also have

$$\sum_E m(\omega_k|E) = c \sum_E m(\omega_k) = 1 \ .$$

Thus,

$$c = \frac{1}{\sum_E m(\omega_k)} = \frac{1}{P(E)} \ .$$

Definition. *The* conditional distribution *given E is the distribution on Ω defined by*

$$m(\omega_k|E) = \frac{m(\omega_k)}{P(E)}$$

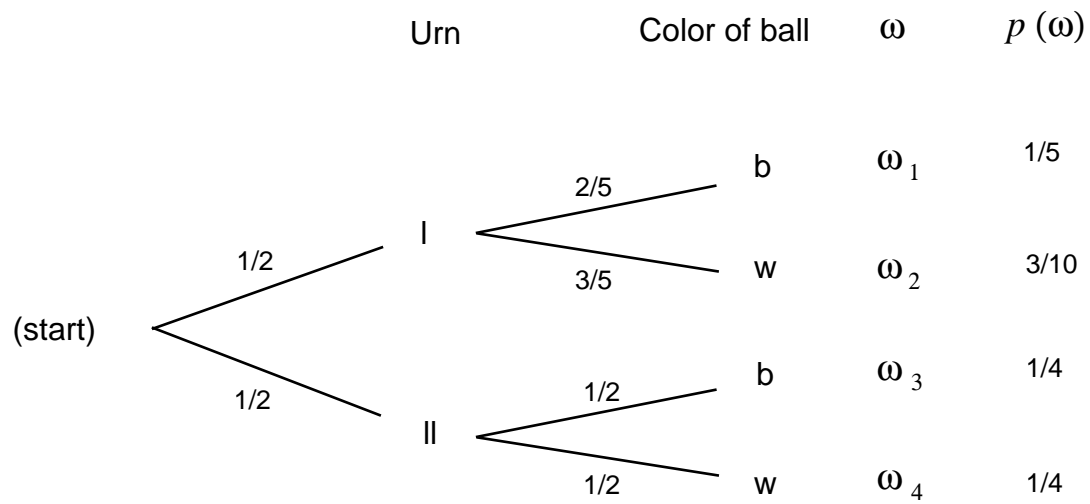*for ω_k in E, and $m(\omega_k|E) = 0$ for ω not in E.*

Then, for a general event F,

$$P(F|E) = \sum_{F \cap E} m(\omega_k|E) = \sum_{F \cap E} \frac{m(\omega_k)}{P(E)} = \frac{P(F \cap E)}{P(E)} \ .$$

We call $P(F|E)$ the *conditional probability of F occurring given that E occurs.*
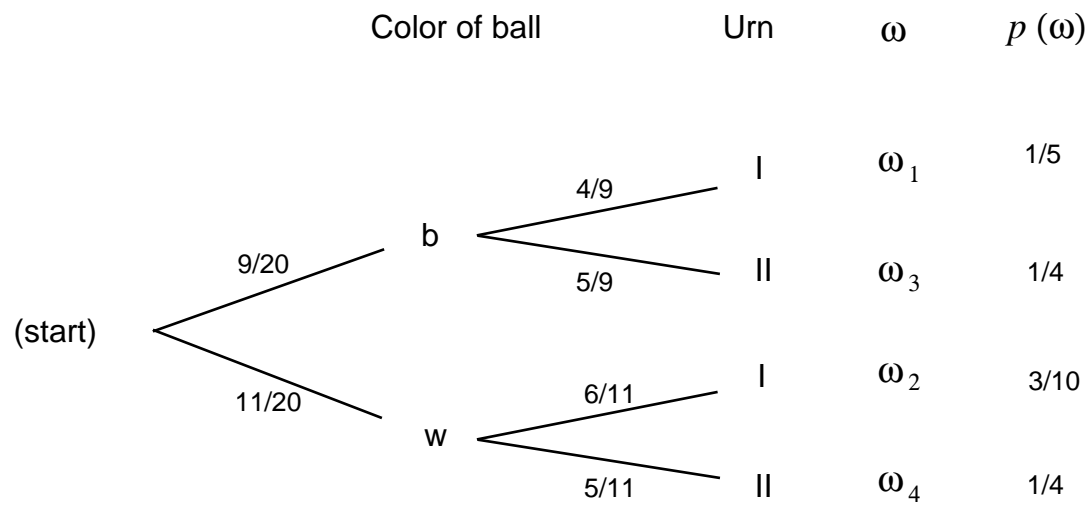
Example

We have two urns, I and II. Urn I contains 2 black balls and 3 white balls. Urn II contains 1 black ball and 1 white ball. An urn is drawn at random and a ball is chosen at random from it. We can represent the sample space of this experiment as the paths through a tree.

Urn   Color of ball ω $p\,(\omega)$

	b	ω_1	1/5

2/5

I

3/5   w ω_2 3/10

1/2

(start)

1/2

II

1/2   b ω_3 1/4

1/2   w ω_4 1/4

- Let B be the event "a black ball is drawn," and I the event "urn I is chosen." Then the branch weight 2/5, which is shown on one branch in the figure, can now be interpreted as the conditional probability $P(B|I)$.

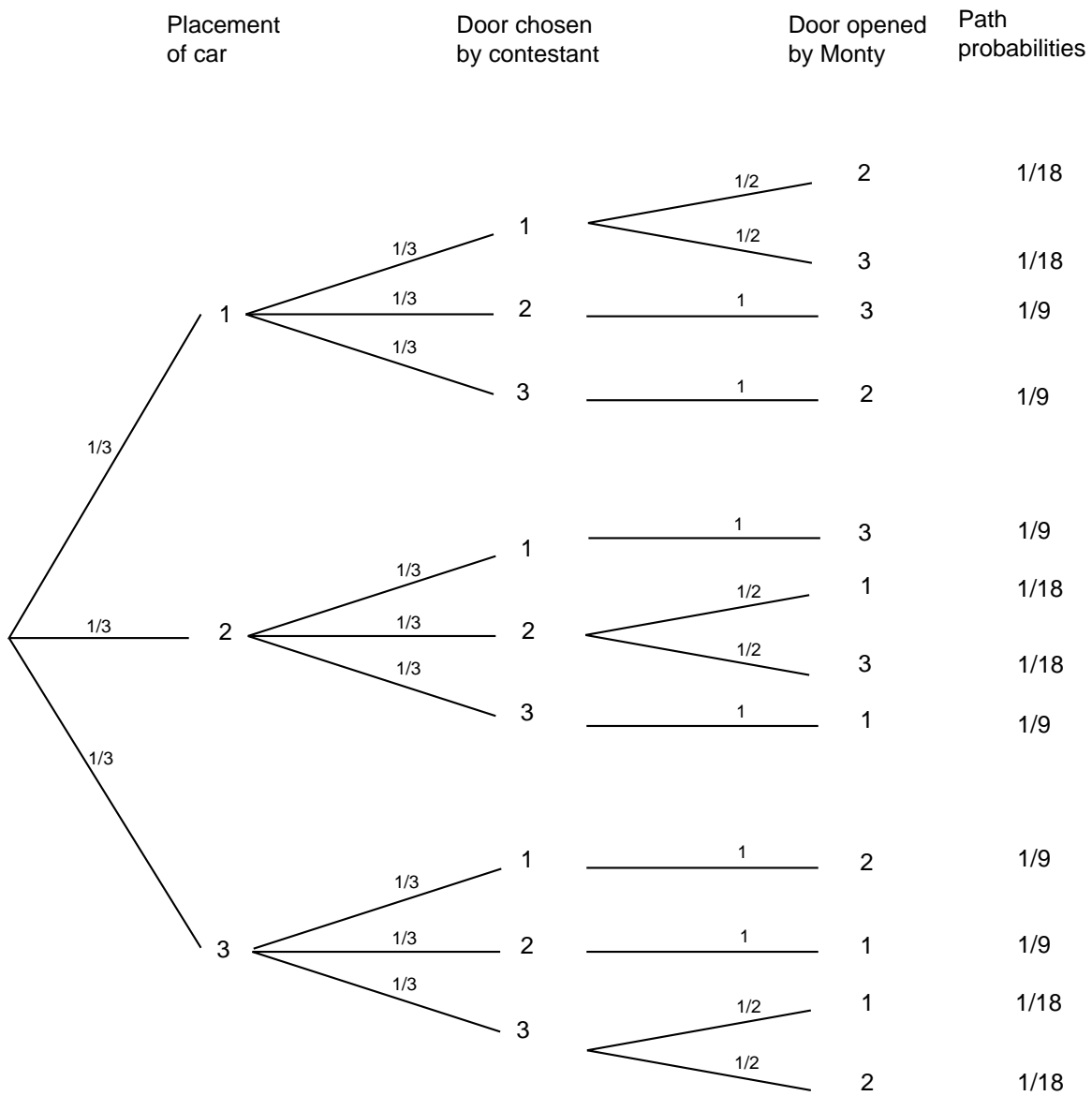- What is $P(I|B)$?

Bayes Probabilities

We have just calculated the *inverse probability* that a particular urn was chosen, given the color of the ball. Such an inverse probability is called a *Bayes probability*.

	Color of ball	Urn	ω	$p\,(\omega)$

Color of ball   Urn ω $p\,(\omega)$

I ω_1 1/5

4/9

b

9/20

5/9   II ω_3 1/4

(start)

I ω_2 3/10

11/20   6/11

w

5/11   II ω_4 1/4

The *Monty Hall* problem

Suppose you're on Monty Hall's *Let's Make a Deal!* You are given the choice of three doors, behind one door is a car, the others, goats. You pick a door, say 1, Monty opens another door, say 3, which has a goat. Monty says to you "Do you want to pick door 2?" Is it to your advantage to switch your choice of doors?

Question: What is the conditional probability that you win if you switch, given that you have chosen door 1 and that Monty has chosen door 3.

Placement of car	Door chosen by contestant	Door opened by Monty	Path probabilities

1/3

1

1/3 1

1/2 2   1/18

1/2 3   1/18

1/3 2

1   3   1/9

1/3 3

1   2   1/9

1/3

2

1/3 1

1   3   1/9

1/3 2

1/2 1   1/18

1/2 3   1/18

1/3 3

1   1   1/9

1/3

3

1/3 1

1   2   1/9

1/3 2

1   1   1/9

1/3 3

1/2 1   1/18

1/2 2   1/18

Problem

Assume that E and F are two events with positive probabilities. Show that if $P(E|F) = P(E)$, then $P(F|E) = P(F)$.

In London, half of the days have some rain. The weather forecaster is correct 2/3 of the time, i.e., the probability that it rains, given that she has predicted rain, and the probability that it does not rain, given that she has predicted that it won't rain, are both equal to 2/3. When rain is forecast, Mr. Pickwick takes his umbrella. When rain is not forecast, he takes it with probability 1/3. Find

1. the probability that Pickwick has no umbrella, given that it rains.

2. the probability that it doesn't rain, given that he brings his umbrella.

Three gamblers, A, B and C, take 12 balls of which 4 are white and 8 black. They play with the rules that the drawer is blindfolded, A is to draw first, then B and then C, the winner to be the one who first draws a white ball. What is the ratio of their chances?

Solve this problem first assuming that the ball is replaced after drawing and then assume that the game is without replacement.

Independent Events

Two events E and F are *independent* if both E and F have positive probability and if

$$P(E|F) = P(E) \ ,$$

and

$$P(F|E) = P(F) \ .$$

Theorem. *If $P(E) > 0$ and $P(F) > 0$, then E and F are independent if and only if*

$$P(E \cap F) = P(E)P(F) \ .$$

Example

Suppose that we have a coin which comes up heads with probability p, and tails with probability q. Now suppose that this coin is tossed twice. Let E be the event that heads turns up on the first toss and F the event that tails turns up on the second toss. Are these independent?

What if A is the event "the first toss is a head" and B is the event "the two outcomes are the same"?

What about I and J, where I is the event "heads on the first toss" and J is the event "two heads turn up."

A set of events $\{A_1, A_2, \ldots, A_n\}$ is said to be *mutually independent* if for any subset $\{A_i, A_j, \ldots, A_m\}$ of these events we have

$$P(A_i \cap A_j \cap \cdots \cap A_m) = P(A_i)P(A_j) \cdots P(A_m),$$

or equivalently, if for any sequence $\bar{A}_1, \bar{A}_2, \ldots, \bar{A}_n$ with $\bar{A}_j = A_j$ or $\tilde{A}_j$,

$$P(\bar{A}_1 \cap \bar{A}_2 \cap \cdots \cap \bar{A}_n) = P(\bar{A}_1)P(\bar{A}_2) \cdots P(\bar{A}_n).$$

Joint Distribution Functions

If we have several random variables $X_1, X_2, \ldots, X_n$ which correspond to a given experiment, then we can consider the joint random variable $\bar{X} = (X_1, X_2, \ldots, X_n)$ defined by taking an outcome ω of the experiment, and writing, as an n-tuple, the corresponding n outcomes for the random variables $X_1, X_2, \ldots, X_n$. Thus, if the random variable X_i has, as its set of possible outcomes the set R_i, then the set of possible outcomes of the joint random variable $\bar{X}$ is the Cartesian product of the R_i's, i.e., the set of all n-tuples of possible outcomes of the X_i's.

Definition

Let $X_1, X_2, \ldots, X_n$ be random variables associated with an experiment. Suppose that the sample space (i.e., the set of possible outcomes) of X_i is the set R_i. Then the joint random variable $\bar{X} = (X_1, X_2, \ldots, X_n)$ is defined to be the random variable whose outcomes consist of ordered n-tuples of outcomes, with the ith coordinate lying in the set R_i. The sample space Ω of $\bar{X}$ is the Cartesian product of the R_i's:

$$\Omega = R_1 \times R_1 \times \cdots \times R_n .$$

The joint distribution function of $\bar{X}$ is the function which gives the probability of each of the outcomes of $\bar{X}$.