Roulette

Math 5 Crew

Department of Mathematics

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Roulette: A Game of Chance

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- INDEPENDENT.

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- $$E(X) = \frac{\#\text{ Ways of Winning}}{\#\text{ Outcomes}}$$
- In Roulette, you will always be purchasing some combination of bets in the from cX for such an X. Though you will never be offered the fair price. Why?

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- Let X be 1 if Red arises and zero otherwise, and note $E(X) = 18/38$.
- The bet we are offered is $2X$ and by the FFMP this bet has a fair price of $E(2X) = 2E(X) = 36/38 = 18/19 < 1$.

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- Suppose for a price of 1 you are offered the bet where you receive 36 if the outcome is the single number 7, and zero otherwise.

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- Let Y be 1 if 7 occurs and zero otherwise, and note $E(Y) = 1/38$.
- Hence our bet $30X$ has a fair price of $E(36Y) = 36E(Y) = 36/38 = 18/19 < 1$.

The Bets

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The Bet	What You Receive	Offered Price	Fair Price
Single Number	$1+35$	1	$18/19$
2 Numbers	$1+17$	1	$18/19$
3 Numbers	$1+11$	1	$18/19$
4 Numbers	$1+8$	1	$18/19$
6 Numbers	$1+5$	1	$18/19$
12 Numbers	$1+2$	1	$18/19$
18 Numbers	$1+1$	1	$18/19$

Expected Value is Constant!

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- We wish to measure how far away from from our expected value we **expect** to be. A good choice of how to do this is the expect value of $(X - E(X))^2$. In other words, we define the *variance* to be

$$V(X) = E((X - E(X))^2).$$

Variance of our examples

- From the FFMT $V(X) = E((X - E(X))^2)$ equals

$$E(X^2) - 2E(X)E(X) + E(X)^2 = E(X^2) - E(X)^2$$

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- For our $2X$ bet, $V(2X) = E(4X^2) - E(2X)^2 = 4(E(X^2) - E(X)^2) = 4(E(X) - E(X)^2$ and by plugging in our $E(X) = 18/38$ equals $4(18/38 - (18/38)^2) = .997$

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- Similarly, for our $30Y$ bet,
$V(30Y) = 36^2(E(Y^2) - E(Y)^2) = 1296(E(Y) - E(Y)^2)$ and by plugging in our $E(Y) = 1/38$ this equals $1296(1/32 - (1/32)^2) = 33.2$

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- we define the *Standard Deviation* as

$$Sd(X) = \sqrt{V(X)}.$$

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The Bet	Fair Price	What You Receive	Var	SDev
Single Number	$18/19$	36	33.21	5.76
2 Numbers	$18/19$	18	16.16	4.02
3 Numbers	$18/19$	12	10.47	3.24
4 Numbers	$18/19$	9	7.63	2.76
6 Numbers	$18/19$	6	4.79	2.19
12 Numbers	$18/19$	3	1.94	1.39
18 Numbers	$18/19$	2	$.997$	$.999$

First Fundamental Theorem of Probability

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$$V(X+Y) = V(X) + V(Y).$$

- Sometimes this is called the Pythagorean Theorem, namely

$$sd(X+Y) = \sqrt{sd(X)^2 + Sd(Y)^2}$$

Proof

By the FFMP

$$V(X+Y) = E((X+Y)^2) - (E(X+Y))^2$$

$$= E(X^2 + Y^2 + 2XY) - (E(X) + E(Y))^2$$

$$= E(X^2) + E(Y^2) + 2E(XY) - (E(X)^2 + E(Y)^2 + 2E(X)E(Y))$$

$$= (E(X^2) - E(X)^2) + (E(Y^2) + E(Y)^2) + 2(E(XY) - E(X)E(Y))$$

$$= V(X) + V(Y) + 2(E(XY) - E(X)E(Y))$$

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As we have seen, $V(X) = E(X^2) - E(X)^2$ and $V(X) = E(Y^2) - E(Y)^2$. The last term, $E(XY) - E(X)E(Y)$, is providing a measurement of how dependent X and Y are. In particular, by the SFMP

$$V(X+Y) = V(X) + V(Y) + 2(E(X)E(Y) - E(X)E(Y)))$$

$$= V(X) + V(Y) + 0$$

What the Casino Sees

- Basically, the casino sees people making N independent bets for some big N. From the casino's view, each bet is nearly in the form $-CX_i$ (were C is determined by the table limit) with (by the FFMP) $E(-CX_i) = -CE(X_i) = -C18/19$ and (by the FFTP) $Sd(CX_i) = CSd(X_i) < C5.76$.

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- Hence, at the end of the day, (by the FFMP) the casino expects to have lost $-CN(18/19)$ with standard deviation (by the FFTP) less than $\sqrt{N}C5.76$, and charged CN dollars in payment.
- Hence the casino has has $CN(1/19)$ dollars with a standard deviation less than of $\sqrt{N}C5.76$.

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- Exercise: Pick a reasonable C and N and use the central limit theorem to estimate the chance that the casino doe not make money from Roulette!