Math 46 Spring 2013

Introduction to Applied Mathematics

Final Exam

Friday, May 31, 3:00-6:00 PM

Your name (please print): Solutions

Instructions: This is a closed book, closed notes exam. Use of calculators is not permitted. You must justify your answers to receive full credit.

The Honor Principle requires that you neither give nor receive any aid on this exam.

Please sign below if you would like your exam to be returned to you in class. By signing, you acknowledge that you are aware of the possibility that your grade may be visible to other students.

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For grader use only:

Problem	Points	Score
1	7	
2	9	
3	8	
4	10	
5	8	
6	8	
7	10	
8	10	
9	10	
Total	80	

1. **[7 points]** In 1940 the Russian applied mathematician A. Kolmogorov assumed there was a law for turbulent fluid flow relating the four quantities: l (length), E (energy, units of ML^2T^{-2}), ρ (density, mass per unit volume), and R (dissipation rate, energy per unit time per unit volume). Using this assumption and the Buckingham Pi Theorem, state the simple form the law must have. Show that there is a (famous!) scaling relation $E = const \cdot l^{\alpha}$ when other parameters are held constant; give α.

$[l] = L \qquad [E] = ML^2T^{-2}$

$[\rho] = ML^{-3} \qquad [R] = \dfrac{[E]}{TL^3} = ML^{-1}T^{-3}$

Dimension Matrix

$$\begin{array}{c c} & \begin{array}{cccc} l & E & \rho & R \end{array} \\ \begin{array}{c} M \\ L \\ T \end{array} & \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 2 & -3 & -1 \\ 0 & -2 & 0 & -3 \end{bmatrix} \end{array}$$

$\Pi_1 = l^a E^b \rho^c R^d$

M: $b + c + d = 0$

L: $a + 2b + -3c - d = 0$

T: $-2b - 3d = 0 \Rightarrow b = -\frac{3}{2}d$

$\Rightarrow c - \frac{1}{2}d = 0 \Rightarrow c = \frac{1}{2}d$

$a - 3d - \frac{3}{2}d - d = 0 \Rightarrow a = \frac{11}{2}d$

let $d = 2$.

$\Pi_1 = l^{11} E^{-3} \rho^{1} R^{2} \Rightarrow E = (\Pi_1 R^{-2} \rho^{-1} l^{11})^{1/3}$

$\Rightarrow \alpha = 11/3$.

2. **[9 points]** Use singular perturbation methods to find a uniform approximate solution to the boundary-value problem

$$\epsilon y'' + 4y' + y = 0, \quad y(0) = 0, \quad y(1) = 1$$

where $0 < \epsilon \ll 1$.

outer solution - leading order equation

$4y' + y = 0$

$\frac{y'}{y} = -1/4 \Rightarrow y(x) = Ce^{-1/4 x}$

Boundary layer at $x=0$.

$y(1) = Ce^{-1/4} = 1 \Rightarrow C = e^{1/4}$

$y(x) = e^{(1-x)/4}$

inner layer $\quad \xi = x/\delta \quad$ Plug into problem.

$\frac{\varepsilon}{\delta^2} Y'' + \frac{4Y'}{\delta} + Y = 0.$

scaling $\frac{\varepsilon}{\delta^2} \sim \frac{1}{\delta} \Rightarrow \delta = O(\varepsilon)$

$\frac{1}{\varepsilon} Y'' + \frac{4Y'}{\varepsilon} + Y = 0$

$Y'' + 4Y' + \varepsilon Y = 0.$

leading order eqn.

$Y'' + 4Y' = 0. \Rightarrow \frac{(Y')'}{Y'} = -4$

$Y' = Ce^{-4\xi}$

$Y(\xi) = Ce^{-4\xi} + D$

use Left BC. $Y(0) = C + D = 0 \Rightarrow C = -D$

$Y_i(\xi) = D \cdot (1 - e^{-4\xi}) \Rightarrow y_i(x) = D(1 - e^{-4(x/\varepsilon)})$

matching condition. $\lim_{\xi \to \infty} Y_i(\xi) = D = \lim_{x \to 0} y_o(x) = e^{1/4}$

Solution is $y(x) = y_i(x) + y_o(x) - D$

3. **[9 points]** Consider the differential equation

$$\frac{d^2y}{dt^2} = -\epsilon\frac{dy}{dt} - 1, \quad y(0) = 0, \quad \frac{dy}{dt}(0) = 1$$

for $\epsilon \ll 1$. Find a two term approximation to the solution.

$y(t) = y_0(t) + \varepsilon y_1(t) + \varepsilon^2 y_2(t) + \cdots$

$y_0'' + \varepsilon y_1'' + \varepsilon^2 y_2'' + \cdots = -\varepsilon(y_0' + \varepsilon y_1' + \varepsilon^2 y_2' + \cdots) - 1$

Collect eqns.

ε^0: $y_0'' = -1 \Rightarrow y_0' = -t + C \Rightarrow y_0(t) = -\frac{t^2}{2} + Ct + D$

$y_0(0) = D = 0$. $y_0'(0) = C = 1$

$y_0(t) = \frac{-t^2}{2} + t$

ε^1: $y_1'' = -y_0' = t - 1$

$\Rightarrow y_1'(t) = \frac{t^2}{2} - t + C$

$y_1(t) = \frac{t^3}{6} - \frac{t^2}{2} + Ct + D$

$y_1(0) = D = 0$ $\quad y_1'(0) = 0 \Rightarrow C$.

2 Term approximation is

$y(t) = -\frac{t^2}{2} + t + \varepsilon\left(\frac{t^3}{6} - \frac{t^2}{2}\right)$

[Bonus 2 points] Find the exact solution and give an expression for the error in the approximation of the solution.

let $v = y'$ $\quad v' = -(1 + v\varepsilon)$ This is separable.

$\frac{v'}{1+\varepsilon v} = -1 \Rightarrow \frac{1}{\varepsilon}\ln(1+\varepsilon v) = -t + C$

$\Rightarrow 1 + \varepsilon v = Ce^{-t\varepsilon}$

$v = \frac{Ce^{-t\varepsilon} - 1}{\varepsilon}$

$$v(t) = \frac{c}{\varepsilon} e^{-t\varepsilon} - \frac{1}{\varepsilon}$$

$$v(0) = \frac{c}{\varepsilon} - \frac{1}{\varepsilon} = 1 \rightarrow c = \varepsilon + 1$$

$$v(t) = \frac{(\varepsilon+1)}{\varepsilon} e^{-t\varepsilon} - \frac{1}{\varepsilon}$$

$$\Rightarrow y(t) = \frac{-(\varepsilon+1)}{\varepsilon^2} e^{-t\varepsilon} - \frac{t}{\varepsilon} + C$$

$$y(0) = -\frac{(\varepsilon+1)}{\varepsilon^2} + C = 0 \rightarrow C = \frac{\varepsilon+1}{\varepsilon^2}$$

$$y_{ex}(t) = \frac{\varepsilon+1}{\varepsilon^2}\left(1 - e^{-t\varepsilon}\right) - \frac{t}{\varepsilon}$$

$$= \frac{\varepsilon+1}{\varepsilon^2}\left(1 - \left(1 + (-t\varepsilon) + \frac{(-t\varepsilon)^2}{2} + \frac{(-t\varepsilon)^3}{3!} + \cdots\right)\right) - \frac{t}{\varepsilon}$$

$$= \frac{\varepsilon+1}{\varepsilon^2}\left(t\varepsilon - \frac{t^2\varepsilon^2}{2} + \frac{t^3\varepsilon^3}{6} - \cdots\right) - \frac{t}{\varepsilon}$$

$$= \left(t - \frac{t^2\varepsilon}{2} + \frac{t^3\varepsilon^2}{6} + \cdots\right) + \left(-\frac{t^2}{2} + \frac{t^3\varepsilon}{6} - \cdots\right)$$

$$\text{Error}(t) = \left|y_{ex}(t) - y_{app}(t)\right| = \left|t - \frac{t^2}{2} + \varepsilon\left(-\frac{t^2}{2} + \frac{t^3}{6}\right) + \frac{t^3\varepsilon^2}{6} + \cdots - (\quad)\right|$$

$$\approx \frac{t^3\varepsilon^2}{6}$$

4. **[10 points]** The reaction-diffusion equation modeling the concentration $u(\boldsymbol{x}, t)$ of a substance in an initially clean body of water $\Omega \subset \mathbb{R}^3$ obeys

$$u_t(\boldsymbol{x}, t) - \Delta u(\boldsymbol{x}, t) + \alpha u(\boldsymbol{x}, t) = f(\boldsymbol{x}, t),\ \boldsymbol{x} \in \Omega,\ t > 0$$
$$u + \frac{\partial u}{\partial n} = 0, \quad \text{on } \partial\Omega$$
$$u(\boldsymbol{x}, 0) = 0, \quad x \in \Omega$$

where f is the pollution source term, and $\alpha > 0$.

(a) Prove that a steady-state (time-independent) solution $u(x)$ to the PDE with given boundary conditions is unique. [Hint: set the t-derivative to zero]

Steady state solution $u_t = 0$.

$$\begin{cases} -\Delta u + \alpha u = f & x \in \Omega \quad t > 0. \\ u = -\frac{du}{dn} & x \in \partial\Omega \end{cases}$$

assume 2 solutions u_1 & u_2. let $w = u_1 - u_2$

The PDE w satisfies is

$$-\Delta w + \alpha w = 0 \quad x \in \Omega$$
$$w = -\frac{dw}{dn} \quad x \in \partial\Omega$$

show $w = 0$.

Use an energy method to

$$\int_\Omega (-\Delta w + \alpha w) w \, dx = 0.$$

Green's 1st identity

$$\hookrightarrow \int_\Omega \nabla w \cdot \nabla w \, dx + \int_\Omega \alpha w^2 \, dx - \int_{\partial\Omega} \frac{dw}{dn} w \, dA = 0$$

$$\int_\Omega (|\nabla w|^2 + \alpha w^2) dx + \int_{\partial\Omega} w^2 \, dA = 0$$

This is a sum of positive items sums to 0. each must be 0.

$$\Rightarrow w = 0.$$

(b) Prove that the time-dependent solution to the full equations above is unique.

Assume u_1 & u_2 are solutions. let $w = u_1 - u_2$

$$\begin{cases} w_t - \Delta w + \alpha w = 0 & x \in \Omega \quad t > 0, \\ w = -\dfrac{\partial w}{\partial n} & x \in \partial\Omega \\ w(x,0) = 0 & x \in \Omega \end{cases}$$

Goal show $w \equiv 0$.

let $E(t) = \int_\Omega w^2 \, dx$ $\qquad E(0) = 0$.

$$E'(t) = \int_\Omega 2 w w_t \, dx$$

$$= 2\int_\Omega w(\Delta w - \alpha w) \, dx$$

$$= 2\left[\int_\Omega w \Delta w \, dx - \alpha \int_\Omega w^2 dx\right]$$

$$= 2\left[\int_\Omega -\nabla w \cdot \nabla w \, dx + \int_{\partial\Omega} w \frac{\partial w}{\partial n} \, dA - \alpha \int_\Omega w^2 dx\right]$$

$$= 2\left[-\int_\Omega |\nabla w|^2 \, dx - \int_{\partial\Omega} w^2 dA - \alpha \int_\Omega w^2 dx\right]$$

≤ 0.

$\Rightarrow E(t)$ is a positive decreasing function.

Since $E(0) = 0$, $\quad E(t) = 0$.

$\Rightarrow w = 0$ //

5. **[8 points]** Use the WKB approximation to estimate the large eigenvalues, λ, of the eigenvalue problem

$$y'' + \frac{\tilde{\lambda} y}{x^2} = 0, \qquad y(1) = 0, \quad y(e) = 0.$$

Rewrite $\frac{1}{\lambda} y'' + \frac{1}{x^2} y = 0$

$k(x) = 1/x$. $\varepsilon = \frac{1}{\lambda}$

This is an oscillator WKB problem. so eigenfunctions come from

$$y(x) = \frac{c_1}{k(x)} \cos\left(\frac{1}{\varepsilon}\int k(x)dx\right) + \frac{c_2}{k(x)} \sin\left(\frac{1}{\varepsilon}\int k(x)dx\right)$$

$$= c_1 x \cos(\lambda \ln x) + c_2 x \sin(\lambda \ln x)$$

$$y(1) = c_1 \cos(\lambda \overset{0}{\ln 1}) + c_2 \sin(\lambda \; 0) = 0 \Rightarrow c_1 = 0.$$

$$y(e) = c_2 e \sin(\lambda) = 0.$$

$$\Rightarrow \lambda = n\pi$$

eigenvalues are $\lambda_n = n\pi$

eigenfunctions are $y_n(x) = x \sin(n\pi \ln x)$

6. **[8 points]** Formulate the integral equation

$$u(t) = 1 + \int_0^t s \ln\left(\frac{s}{t}\right) u(s)ds \;=\; 1 + \int_0^t s\big(\ln(s) - \ln(t)\big)\, u(s)ds.$$

as an initial value problem. Show how you get the initial condition(s).

take derivative

$$u'(t) = \int_0^t \frac{-s u(s)}{t}\, ds + \underbrace{t(\ln(t) - \ln(t))u(t)}_{0}$$

$$u''(t) = \frac{-1}{t} t u(t) + \frac{-1}{t^2}\int_0^t s u(s)$$

$$= -u(t) - \frac{1}{t}u'(t)$$

$$\lim_{t\to 0} \frac{\int_0^t s\,u(s)\,ds}{t} \overset{LH}{=} \lim_{t\to 0} \frac{t\,u(t)}{0} = 0.$$

$\Rightarrow u'(0) = 0.$

$u(0) = 1$

For intial condition.

$$\begin{cases} u'' + \frac{1}{t}u'(t) + u(t) = 0 \\ u'(0) = 0 \\ u(0) = 1 \end{cases}$$

7. **[10 points]** In the half plane $\{(x, y) : y \geq 0\}$, there is a stationary and bounded temperature distribution u. (ie. solution is independent of time, so it satisfies

$$u_{xx} + u_{yy} = 0, \quad x \in \mathbb{R},\, y > 0$$

and u is bounded for $y > 0$.) It is known that $u(x, 0) = \frac{1}{1+x^2}$. Determine $u(x, y)$ for all $y > 0$. (Do not forget to explain where the $y \to \infty$ condition enters.)

Take Fourier transform wrt x.

$(-i\xi)^2 \hat{u} + \hat{u}_{yy} = 0. \Rightarrow \hat{u}_{yy} - \xi^2 \hat{u} = 0.$

$\Rightarrow \hat{u}(\xi, y) = C_1(\xi) e^{-|\xi| y}$

So we have bounded solutions.

BC. in fourier space $\hat{u}(\xi, 0) = \mathcal{F}\left(\frac{1}{1+x^2}\right) = \pi e^{-|\xi|}$

So $\hat{u}(\xi, 0) = \pi e^{-|\xi|} = C_1(\xi)$.

$\Rightarrow \hat{u}(\xi, y) = \pi e^{-|\xi|(1+y)}$

Tranforming back $\Rightarrow u(x, y) = \dfrac{1+y}{(1+y)^2 + x^2}$

8. **[10 points]** Consider the differential operator

$$Au = u''(x), \quad 0 < x < \pi,$$

with boundary conditions $u(0) = u'(\pi) = 0$.

(a) Prove via an energy method that the eigenvalues of A have a strict sign. What is the sign?

$$Au = \lambda u \Rightarrow \int_0^{\pi} u'' u \, dx = \lambda \int_0^{\pi} u^2 dx$$

$$\underbrace{u'u \Big|_0^{\pi}}_{0 \text{ by BC.}} - \int_0^{\pi} (u')^2 dx = \lambda \int_0^{\pi} u^2 dx$$

$$\Rightarrow \lambda \leq 0.$$

(b) Find the eigenvalues and construct the complete orthogonal set of functions in $L^2(0, \pi)$ from which all solutions can be written in series form.

eigenvalues satisfy $\begin{cases} u'' - \lambda u = 0. \\ u(0) = u'(\pi) = 0. \end{cases}$

$u(x) = C_1 \cos(\sqrt{\lambda} x) + C_2 \sin(\sqrt{\lambda} x)$

$u(0) = 0 = C_1$

$u'(x) = C_2 \sqrt{\lambda} \cos(\sqrt{\lambda} x)$

$u'(0) = C_2 \sqrt{\lambda} \cos(\sqrt{\lambda} \pi) = 0 \Rightarrow \sqrt{\lambda}\pi = \frac{\pm(n\pi)}{2}$. n-odd.

$\Rightarrow \lambda = -\frac{n^2}{4}$

$\Rightarrow$ eigenvalues are $\lambda_n = \frac{-(2n+1)^2}{4}$

eigenfunctions are $u_n(x) = \sin\left(\frac{2n+1}{2} x\right)$

(c) Does the equation $Au(x) = f(x)$ have a Green's function? If so, give an expression for it. If not, explain why. (Recall the Green's function can be expressed as a series.)

$\lambda = 0$ is not an eigenvalue of A thus there is a Greens function.

2 options
- derive via formula. ....
- use series. representation.. → much easier.

$$g(x,\xi) = \frac{2}{\pi}\sum_{n=0}^{\infty} \frac{\sin\left(\frac{2n+1}{2}x\right)\sin\left(\frac{2n+1}{2}\xi\right)}{\lambda_n}$$

↑ normalize

(d) Use the Greens function, or if not possible, another ODE solution method, to write an explicit formula for the solution $u(x)$ to $Au(x) = f(x)$ with the above boundary conditions, in terms of a general driving $f(x)$.

The solution $u(x) = \int_0^{\pi} g(x,\xi) f(\xi)\, d\xi$.

9. **[10 points]**

(a) Is the sequence $f_n(x) = \frac{\sin(nx)}{n}$, $n = 1, 2, \ldots$ converge to the zero function on $(-\pi, \pi)$ pointwise? uniformly? in L^2?

Ptwise.

Fix x

$$\lim_{n\to\infty} \left|\frac{\sin(nx)}{n}\right| \leq \lim_{n\to\infty} \frac{1}{n} = 0.$$

converges ptwise & uniformly.

$$\lim_{n\to\infty} \int_{-\pi}^{\pi} \frac{\sin^2(nx)}{n^2}\,dx = \lim_{n\to\infty}\int_{-\pi}^{\pi} \frac{1}{2n^2}\left(1-\cos(2nx)\right)dx$$

$$= \lim_{n\to\infty} \frac{1}{2n^2}\left(x - \frac{\sin(2nx)}{2n}\right)\Big|_{-\pi}^{\pi} = \lim_{n\to\infty} \frac{\pi}{n^2} = 0.$$

$\Rightarrow$ Converges in all

(b) Consider the functions $f(x) = x^3 e^{-x^4}$ and

$$u(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & \text{otherwise.} \end{cases}$$

$0 < x - y < 1$

$-x < -y < 1 - x$

$x - 1 < y < x.$

Compute the convolution of f and u.

$$(f * u)(x) = \int_{-\infty}^{\infty} f(x-y)\,u(y)\,dy =$$

$$= \int_{x-1}^{x} y^3 e^{-y^4}\,dy$$

$u = -y^4$

$du = -4y^3\,dy$

$$= \frac{-1}{4} e^{-y^4}\Big|_{x-1}^{x} = \frac{-1}{4}\left(e^{-x^4} - e^{-(x-1)^4}\right)$$

(c) Compute the Fourier transform of

$$f(t) = \begin{cases} 1 - |t|, & |t| < 1 \\ 0, & \text{otherwise.} \end{cases}$$

$$\mathcal{F}(f) = \int_{\infty}^{\infty} e^{i\xi t} f(t)\,dt = \int_{-1}^{0} (1+t)e^{i\xi t}\,dt + \int_{0}^{1} (1-t)e^{i\xi t}\,dt$$

$u = 1+t$

$dv = e^{i\xi t}dt$

$$= \frac{(1+t)}{i\xi} e^{i\xi t}\Big|_{-1}^{0} - \int_{-1}^{0} \frac{e^{i\xi t}}{i\xi}\,dt + \frac{(1-t)e^{i\xi t}}{i\xi}\Big|_{0}^{1} + \int_{0}^{1} \frac{e^{i\xi t}}{i\xi}\,dt$$

$$= \frac{1}{i\xi} - 0 - \frac{1}{(i\xi)^2}\left(1 - e^{-i\xi}\right) + \frac{0-1}{i\xi} + \frac{1}{(i\xi)^2}\left(e^{i\xi} - 1\right) = \frac{2 - e^{-i\xi} - e^{i\xi}}{\xi^2}$$

(d) Is $\cos(x) - 1 + \frac{x^2}{2} = O(x^4)$ hold as $x \to 0$? Prove your answer.

$$\left| \frac{\cos(x) - 1 + \frac{x^2}{2}}{x^4} \right| = \left| \frac{\sum_{j=0}^{\infty} \frac{(-1)^{j+1} x^{2j}}{(2j)!} - 1 + \frac{x^2}{2}}{x^4} \right| = \left| \frac{\frac{x^4}{4!} - \frac{x^5}{5!}}{x^4} \right| \leq \frac{1}{4!}$$

$$\Rightarrow \cos(x) - 1 + \frac{x^2}{2} \text{ is } O(x^4)$$

(e) **[Bonus 2 points]** Place the following four terms in the *correct* order to form an asymptotic series as $\epsilon \to 0$: $f(\epsilon) \sim \epsilon^{1/2} + \epsilon \ln \epsilon + \epsilon^3 + \epsilon^{-1} + \ldots$

ε^{-1} - biggest.

$$\varepsilon^{-1} + \varepsilon^{1/2} + \varepsilon \ln \varepsilon + \varepsilon^3$$

Useful formulae

non-oscillatory WKB approximation

$$y(x) = \frac{1}{\sqrt{k(x)}} e^{\pm\frac{1}{\epsilon}\int k(x)dx}$$

Binomial series

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

Euler relations

$$e^{i\theta} = \cos\theta + i\sin\theta, \quad \cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}, \quad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2}$$

Power-reduction identities

$$\begin{aligned}
\cos^3\theta &= \frac{1}{4}(3\cos\theta + \cos 3\theta) \\
\cos^2\theta\sin\theta &= \frac{1}{4}(\sin\theta + \sin 3\theta) \\
\cos\theta\sin^2\theta &= \frac{1}{4}(\cos\theta - \cos 3\theta) \\
\sin^3\theta &= \frac{1}{4}(3\sin\theta - \sin 3\theta)
\end{aligned}$$

Addition formulae

$$\begin{aligned}
\sin(\alpha\pm\beta) &= \sin\alpha\cos\beta \pm \cos\alpha\sin\beta \\
\cos(\alpha\pm\beta) &= \cos\alpha\cos\beta \mp \sin\alpha\sin\beta
\end{aligned}$$

Leibniz's rule

$$\frac{d}{dx}\left(\int_{a(x)}^{b(x)} f(x,y)\,dy\right) = \left(\int_{a(x)}^{b(x)} f_x(x,y)\,dy\right) + f(x,b(x))b'(x) - f(x,a(x))a'(x)$$

Fourier Transforms:
$\hat{u}(\xi) = \int_{-\infty}^{\infty} e^{i\xi x}u(x)dx$
$u(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\xi x}\hat{u}(\xi)d\xi$

Fourier Transform table

$u(x)$	$\hat{u}(\xi)$
$\delta(x-a)$	$e^{ia\xi}$
e^{ikx}	$2\pi\delta(k+\xi)$
e^{-ax^2}	$\sqrt{\frac{\pi}{a}}e^{-\frac{\xi^2}{4a}}$
$e^{-a\|x\|}$	$\frac{2a}{a^2+\xi^2}$
$H(a-\|x\|)$	$\frac{2\sin(a\xi)}{\xi}$
$u^{(n)}(x)$	$(-i\xi)^n\hat{u}(\xi)$
$u * v$	$\hat{u}(\xi)\hat{v}(\xi)$

v not u.

where $H(x) = \begin{cases} 1 & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$

Green's first identity: $\int_\Omega (u\Delta v + \nabla u \cdot \nabla v)\, dx = \int_{\partial\Omega} u \frac{\partial u}{\partial n} dA$

Product rule for divergence: $\nabla \cdot (u\boldsymbol{J}) = u\nabla \cdot \boldsymbol{J} + \boldsymbol{J} \cdot \nabla u$