Degenerate Fredholm equations — notes & worked examples. [corrected]

Want to solve: $\boxed{Ku - \lambda u = f}$ 2nd kind (if $\lambda=0$, called 1st kind)

First, why do we care about eigenvalues of operator K?

- if u is solution to $Ku = f$, then can add any solution to $Ku=0$ and still have a solution.

  $\Rightarrow$ $\mathrm{Nul}\, K := \{u : Ku = 0\}$ tells us the non-uniqueness of soln. ~ this is the $\lambda=0$ eigenspace of K.

- if u is soln to $Ku - \lambda u = f$, it's unique unless λ = eigenval of K, and then can add any amount of eigenfunction to u, still a solution.

note λ-eigenspace of K $:= \mathrm{Nul}\,(K - \lambda I)$

All this is same as you already know from lin. alg. (Math 22):

$A\vec{x} = \vec{b}$, unique soln if $\mathrm{Nul}\, A = \{\vec{0}\}$, ie $\det A \neq 0$, A invertible.

otherwise if $\vec{x}$ is a soln, then $\vec{x} + \vec{v}$ is too, $\forall \vec{v} \in \mathrm{Nul}\, A$

$A\vec{x} - \lambda\vec{x} = \vec{b}$, unique soln. unless λ = eigval of A ⟸ Please review lin. alg. if unclear

Eg. $(Ku)(x) = \int_0^1 \overbrace{(\sin \pi x)\, y}^{k(x,y)}\, u(y)\, dy$ is degen. since $k(x,y) = \alpha_1(x)\,\beta_1(y)$, $n=1$ with funcs $\begin{cases}\alpha_1(x) = \sin \pi x \\ \beta_1(x) = x\end{cases}$

Solve on $[0,1]$:

i) $Ku(x) = 3x$
ii) $Ku(x) = 3\sin \pi x$ } 1st kind
iii) $Ku(x) - u(x) = 1$
iv) $Ku(x) - \frac{1}{\pi}u(x) = 1$
v) $Ku(x) - \frac{1}{\pi}u(x) = \frac{2}{3} - x^2$ } 2nd kind

The first two we can do using basic calculus:

i) $\int_0^1 \sin \pi x \; y\, u(y)\, dy = 3x$ has no solution since LHS is multiple of $\sin \pi x$ but RHS isn't.

an example of: $f \notin \mathrm{Span}\{\alpha_j\}$ $\Rightarrow$ no soln. (p.238)

ii) $\sin \pi x \int_0^1 y\, u(y)\, dy = 3 \sin \pi x$ (note we moved it out) any u s.t. $\int_0^1 y\, u(y)\, dy = 3$ is soln.

a const such soln. is $u(x) = 6$ (check it) so general soln is $u(x) = 6 + v(x)$ for any $v(x)$ s.t. $(v, x) = 0$.

Answers to i) & ii) reflected the following : K has ∞-multiplicity zero-eigenvalue with eigenspace consisting of all funcs v orthog to $\beta_1(x) = x$.

- What is rest of spectrum of K ? Just that of matrix A, with elements (β_i, α_j).

$n=1$ so A is 1-by-1 : $A = [(\beta_1, \alpha_1)] = [(\sin\pi x, x)] = [\int_0^1 x \sin\pi x \, dx] = [1/\pi]$

A has n eigenvalues (counting multiplicities), ie the single eigenvalue $\lambda_1 = 1/\pi$

Armed with that, we can solve 2nd kind:

iii) $Ku - u = 1$ — $\lambda = 1$ is not eigenvalue of K, so there's a unique soln.

Convert to matrix (1-by-1) problem: $A\vec{c} - \vec{c} = \vec{f}$ — col. vec entries $(\beta_1, f) = (x, 1) = 1$

ie $[\frac{1}{\pi}]c - c = 1$

$\Rightarrow c = \dfrac{1}{\frac{1}{\pi} - 1}$

Eqn (*) from lec (= book eqn (4.31)) gives $\lambda u(x) = f(x) - \sum_{j=1}^{n} c_j \alpha_j(x)$, $\lambda = 1$

ie $u(x) = 1 - c\alpha_1(x)$

$= 1 - \dfrac{1}{\frac{1}{\pi} - 1} \sin\pi x$ soln.

iv) $Ku - \frac{1}{\pi}u = 1$ — λ is an eigval of K.

$\Rightarrow$ soln. only if $\vec{f}$ in range of $A - \lambda I$ matrix.

But here $A - \lambda I = [\frac{1}{\pi}] - \frac{1}{\pi}[1] = [0]$

and $\vec{f} = [1]$ as before, so no soln.

v) $Ku - \frac{1}{\pi}u = \frac{2}{3} - x^2$ — same as above except $\vec{f} = [(x, \frac{2}{3} - x^2)] = [0]$ by Fourier sine orthog. on $(0,1)$

so $\vec{f}$ is in $\mathrm{Ran}(A - \lambda I)$

lin. alg: $A\vec{c} - \frac{1}{\pi}\vec{c} = \vec{f}$ gives $0 . c = 0$ so $c =$ any real number

Use (*) : $\lambda u(x) = f(x) - c\alpha_1(x)$

ie $u(x) = \pi\left[\frac{2}{3} - x^2 - c\sin\pi x\right]$ for any $c \in \mathbb{R}$, general soln

This covers all the cases you need. For the $n=2$ case, see Ex 4.15 (p.239).