Math 23 Final Exam

June 5, 2015

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Wilder Room 111

PRINT NAME: ____

Instructions: This is a closed book, closed notes exam. **Use of calculators is not permitted. You must justify all of your answers to receive credit. A correct answer with no justification will receive minimal credit.**

You have **three hours** to work on this exam.

Circle your instructor's name above.

The **Honor Principle** requires that you neither give nor receive any aid on this exam.

1. ________ /10

2. ________ /10

3. ________ /10

4. ________ /10

5. ________ /10

6. ________ /10

7. ________ /10

8. ________ /10

9. ________ /10

10. ________ /10

11. ________ /10

12. ________ /10

13. ________ /10

14. ________ /10

Total: ________ /140

1. (10 points) Solve the initial value problem

$$y' = 2y^2 + xy^2, \qquad y(0) = 1.$$

2. (10 points) Determine (without solving the problem) the interval in which the solution of the initial value problem

$$y' + (\tan t)y = \sin t, \qquad y(\pi) = 0.$$

is certain to exist.

3. (10 points) Let y_1 and y_2 be solutions of a differential equation $y'' + p(t)y' + q(t)y = 0$, where p and q are continuous functions. Prove that if y_1 and y_2 have maxima or minima at the same point, then they cannot be a fundamental set of solutions.

4. (10 points) Find the cosine Fourier series for f with period $2L = 4$:

$$f(x) = \begin{cases} 1, & \text{if } 0 < x < 1 \\ 0, & \text{if } 1 < x < 2 \end{cases}$$

5. (10 points) Find the steady state solution of the heat conduction equation

$$\alpha^2 u_{xx} = u_t, \qquad u(0,t) = 30, u(40,t) = -20$$

(you do not need to find the transient solution!).

6. (10 points) Suppose that $\mathbf{x} = \mathbf{x}^{(0)}$ is a solution of $\mathbf{A}\mathbf{x} = \mathbf{b}$. Show that if ξ is a solution of $\mathbf{A}\xi = \mathbf{0}$ and α is any constant, then $\mathbf{x} = \mathbf{x}^{(\mathbf{0})} + \alpha\xi$ is also a solution of $\mathbf{A}\mathbf{x} = \mathbf{b}$.

7. (10 points) Consider the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = y(y-1)(y-2).$$

(a) Determine the critical points (equilibria).

(b) Draw the phase line.

(c) Determine whether each equilibrium is stable or unstable.

8. (10 points) Find the solution to the heat conduction problem

$$\begin{array}{rll} 100u_{xx} = u_t & 0 < x < 1 & t > 0; \\ u(0,t) = 0, \quad u(1,t) = 0, & & t > 0; \\ u(x,0) = 2\sin(2\pi x) - \sin(5\pi x) & 0 \le x \le 1. & \end{array}$$

9. (10 points) Consider an elastic string of length $L = 8$ whose ends are held fixed. The string is set in motion with no initial velocity from an initial position $u(x, 0) = f(x)$. Find the displacement $u(x, t)$ for the initial position $(0 < x < 8)$

$$f(x) = \begin{cases} 1, & 3 < x < 5 \\ 0, & \text{otherwise.} \end{cases}$$

10. (10 points) Express the general solution of the system of equations

$$\begin{aligned} x_1' &= x_1 + x_2 + x_3 \\ x_2' &= 2x_1 + x_2 - x_3 \\ x_3' &= -x_2 + x_3 \end{aligned}$$

in terms of real-valued functions (Hint: the eigenvalues are $\lambda_1 = -1$ and $\lambda_1 = \lambda_2 = 2$).

11. (10 points) Find a fundamental set of solutions y_1 and y_2 of

$$y'' - xy' - y = 0, \qquad x_0 = 0.$$

12. (10 points) Let f be a continuous function on $[0, L]$. Show that there is a Fourier series converging to f of the form

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{(2n-1)\pi x}{2L},$$

where

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{(2n-1)\pi x}{2L} dx.$$

Hint: first extend f into $(L, 2L)$ so that $f(2L - x) = f(x)$ for all $x \in [0, L]$. Let the resulting function be extended into $(-2L, 0)$ as an odd function and elsewhere as a periodic function of period $4L$.

13. (10 points) Consider a uniform bar of length L having an initial temperature distribution given by $f(x)$, $x \in [0, L]$. Assume that the temperature at the end $x = 0$ is held at 0, while the end $x = L$ is insulated so that no heat passes through it. Show that the fundamental solutions of the partial differential equation and boundary conditions are

$$u_n = e^{-(2n-1)^2\pi^2\alpha^2 t/4L^2} \sin(2n-1)\pi x/2L, \qquad n = 1, 2, 3, ...$$

14. (10 points) The heat conduction equation in two space dimensions is $\alpha^2(u_{xx} + u_{yy}) = u_t$. Assuming that $u(x, y, t) = X(x)Y(y)T(t)$, find ordinary differential equations that are satisfied by $X(x), Y(y)$, and $T(t)$.