Markov Chains

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- Most of our study of probability has dealt with independent trials processes.

- A Markov chain is aprocess in which the outcome of a given experiment can affect the outcome of the next experiment.

Markov Chains

- We have a set of *states*, $S = \{s_1, s_2, \ldots, s_r\}$.
- The process starts in one of these states and moves successively from one state to another.
- Each move is called a *step*.
- If the chain is currently in state s_i, then it moves to state s_j at the next step with a probability denoted by p_{ij}, and this probability does not depend upon which states the chain was in before the current state.

- The probabilities p_{ij} are called transition probabilities.

- The process can remain in the state it is in, and this occurs with probability p_{ii}.

- An initial probability distribution, defined on S, specifies the starting state.

Example

According to Kemeny, Snell, and Thompson, the Land of Oz is blessed by many things, but not by good weather. They never have two nice days in a row. If they have a nice day, they are just as likely to have snow as rain the next day. If they have snow or rain, they have an even chance of having the same the next day. If there is change from snow or rain, only half of the time is this a change to a nice day. We take as states the kinds of weather R, N, and S.

$$P = \begin{pmatrix} 1/2 & 1/4 & 1/4 \\ 1/2 & 0 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{pmatrix}.$$

Transition Matrix

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- We denote this probability by $p_{ij}^{(2)}$.

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$$p_{13}^{(2)} = p_{11}p_{13} + p_{12}p_{23} + p_{13}p_{33} \; .$$

Theorem. *Let P be the transition matrix of a Markov chain. The ijth entry $p_{ij}^{(n)}$ of the matrix P^n gives the probability that the Markov chain, starting in state s_i, will be in state s_j after n steps.*

- Consider the long-term behavior of a Markov chain when it starts in a state chosen by a probability distribution on the set of states, which we will call a *probability vector*.

- A probability vector with r components is a row vector whose entries are non-negative and sum to 1.

- If u is a probability vector which represents the initial state of a Markov chain, then we think of the ith component of u as representing the probability that the chain starts in state s_i.

Theorem. *Let P be the transition matrix of a Markov chain, and let u be the probability vector which represents the starting distribution. Then the probability that the chain is in state s_i after n steps is the ith entry in the vector*

$$u^{(n)} = uP^n \ .$$

Example

- In the Land of Oz example let the initial probability vector u equal $(1/3, 1/3, 1/3)$.

- What is the distribution of the states after three days?