More Model Theory Notes

Miscellaneous information, loosely organized.

1. Kinds of Models

A countable homogeneous model $\mathcal{M}$ is one such that, for any partial elementary map $f : A \to M$ with $A \subseteq M$ finite, and any $a \in M$, f extends to a partial elementary map $A \cup \{a\} \to M$. As a consequence, any partial elementary map to M is extendible to an automorphism of $\mathcal{M}$. Atomic models (see below) are homogeneous.

A prime model of T is one that elementarily embeds into every other model of T of the same cardinality. Any theory with fewer than continuum-many types has a prime model, and if a theory has a prime model, it is unique up to isomorphism. Prime models are homogeneous. On the other end, a model is *universal* if every other model of its size elementarily embeds into it.

Recall a *type* is a set of formulas with the same tuple of free variables; generally to be called a type we require consistency. The type of an element or tuple from a model is all the formulas it satisfies. One might think of the type of an element as a sort of identity card for automorphisms: automorphisms of a model preserve types. A complete type is the analogue of a complete theory, one where every formula of the appropriate free variables or its negation appears. Types of elements and tuples are always complete. A type is *principal* if there is one formula in the type that implies all the rest; principal complete types are often called *isolated*. A trivial example of an isolated type is that generated by the formula $x = c$ where c is any constant in the language, or $x = t(\bar{c})$ where t is any composition of appropriate-arity functions and $\bar{c}$ is a tuple of constants. In the language of algebraically closed fields, the formula $p(x) = 0$, for p a polynomial with coefficients in $\mathbb{Q}$, isolates a type – complete types may hold of multiple elements.

Omitting types theorem (Henkin, Keisler). If T is a countable, complete theory and $\Gamma(\bar{x})$ is a type that is not principal with respect to T, then T has a model that omits Γ (i.e., does not realize it).

An atomic model is an extreme example of this: it omits all non-principal types. If the types of elements of models of a theory T are always principal (a.k.a. *isolated*), T is $\aleph_0$-categorical, and conversely (Ryll-Nardjewski). If $\mathcal{M}$ and $\mathcal{N}$ are both countable atomic models of T they must be isomorphic. If the complete theory T has infinite models, then a countable model of T is prime if and only if it is atomic. The prime to atomic direction boils down to the omitting types theorem: a model can't embed into anything that realizes more types that it does. The atomic to prime direction is a little more complicated. Note that the restriction to countability is important throughout this paragraph!

At the opposite end is a saturated model: a countable model $\mathcal{M}$ of a complete theory T is saturated if every $\mathcal{M}$-consistent 1-type $\Gamma(x)$ with parameters from M is realized in $\mathcal{M}$. A complete theory T has a countably saturated model if and only if for every $n \in \mathbb{N}$, T has countably many n-types. If $\mathcal{M}$ and $\mathcal{N}$ are both countable saturated models of T they must be isomorphic (this holds also for larger cardinalities). A corollary to those two results is that any complete theory with only countably many nonisomorphic countable models must have a countable saturated model. In contrast to the prime/atomic connection, universal models are not necessarily saturated. Saturated models are those that are universal and homogeneous.

Examples: an atomic model may be built from any other model by extending the language and theory to name each element of the original model by a constant. Any model of a categorical theory is prime, atomic, and saturated. Viewing $\mathbb{R}$ and the field of algebraic numbers over $\mathbb{Q}$ as models of RCF (see below), $\mathbb{R}$ is saturated and the algebraic numbers are atomic.

RCF is the theory of real closed fields, in the language of fields. It contains all the field axioms, as well as the following:

- For each n, the formula $\forall x_1 \dots x_n (x_1^2 + \dots + x_n^2 \neq -1)$, distinguishing positive and negative values.
- Closure under square root: $\forall x \exists y (x = y^2 \ \vee \ -x = y^2)$.
- For each *odd* n, the formula $\forall x_1 \dots x_n \exists y (y^n + x_1 y^{n-1} + \dots + x_{n-1} y + x_n = 0)$, closure in the sense of every odd-degree polynomial having a real root.

More on all of this in §3.

2. Size of Models

Löwenheim-Skolem Theorem. if a countable theory T has an infinite model, then it has models of all infinite cardinalities.

This is usually split into "downward" Löwenheim-Skolem, that every consistent theory in a countable language has a countable or finite model, and "upward", that if a theory has a model of infinite cardinality κ, then it has models of every cardinality $\geq \kappa$. In fact the downward version generalizes to say every consistent theory has a model of cardinality at most that of the language. The two together give the more general full result that any T with an infinite model has models of all infinite cardinalities greater than or equal to that of its language.

The proof of the downward version in the countable case is a *Henkin construction*, where the language is expanded by countably many constants called witnesses and the theory by formulas giving specific witnesses for existential formulas: $\exists v \varphi(v) \rightarrow \varphi(b)$ for appropriate b. Care is taken to maintain consistency throughout. Then, in a step reminiscent of the application of Zorn's lemma to find a maximal algebraic closure of a field, we find a maximally consistent expansion of our original theory. "Maximal" here means complete. We can explicitly build a model for this expansion, letting its elements be the witnesses (modded out by equivalence); this model with the constants removed from the language (the names for the elements erased) is a model of our original theory. This is exactly the proof that consistency implies satisfiability in the completeness theorem, also.

Upward Löwenheim-Skolem is more straightforward. Just as in the proof that nonstandard models exist, we add a collection of sentences that collectively are only satisfiable by a "bigger" model (one that is larger cardinality, rather than one that has nonstandard elements) but for which finite subsets are satisfiable within a model of the original theory. Compactness does the rest. This time we add as many constants as the cardinality we want, and sentences that say each pair of distinct constants are nonequal.

3. Number of Models

The theory of dense linear orders without endpoints is countably categorial: $\mathbb{Q}$ is it. However, if you add in an infinite set of constants c_i, $i \in \mathbb{N}$, and axioms saying $c_i < c_{i+1}$ for each i, you get a theory T (an *Ehrenfeucht theory*) that is still complete, but not categorical. In fact, T has exactly three nonisomorphic models. In the first (call it $\mathcal{A}$), the sequence c_i is cofinal in the model: it has no upper bound. In the second (call it $\mathcal{B}$), the sequence has an upper bound but not a least upper bound (it "converges to an irrational"). In the third (call it $\mathcal{C}$), the sequence has a least upper bound (it "converges to a rational"). We can't get at the distinction in a first-order way: we can't quantify over all the constants, and so we have no way to specify anything other than the sets $\{c_i\}$, $(-\infty, c_i)$, (c_i, ∞), and sets we build from those via unions, intersections, and complements.

However, "upper bound of the c_i" is a perfectly good *type*; it is just that it is a *necessarily infinite* set of first-order formulas. However, "least upper bound of the c_1" is *not* a type because it is not expressible in a first-order way.

Model $\mathcal{A}$ is atomic and prime, realizing only the finite types $x < c_1$, $x = c_i$, and $c_i < x < c_{i+1}$. Model $\mathcal{B}$ is saturated. Model $\mathcal{C}$ is not atomic or prime, and also not saturated, since with a parameter b for the element to which the c_i converge, we get the unrealized type $\{c_i < x < b : i \in \mathbb{N}\}$. However, it is universal; model $\mathcal{B}$ may be embedded elementarily in pieces, the part below the upper bound below b and the part above the upper bound above b, skipping some interval $[b, d]$ in between. Model $\mathcal{C}$ fails homogeneity; we should have an automorphism from b (though without the name b)) onto any other point greater than all the c_i, but that cannot be done without either failing bijectivity or reversing order with another upper bound, which is not allowed in an automorphism.

Similar tricks may be played to get theories with exactly 4, 5, 6, ... non-isomorphic models. However, exactly two nonisomorphic models is impossible (Vaught): if T has only countably many countable models, one must be saturated. Any complete theory with a countable saturated model also has a countable atomic model. If the saturated and atomic models are the same, then T is countably categorical. If not, the saturated model $\mathcal{M}$ realizes some non-principal type, $\mathrm{tp}(\bar{b})$ for some $\bar{b}$ a tuple from M. Expand $\mathcal{L}$ by adding constants for each of the elements of $\bar{b}$, and expand T to $T' = \mathrm{Th}(\mathcal{M})$ in the new language. The sketch continues by using saturation of $\mathcal{M}$ to show T' can have only one model up to isomorphism, and then use the non-categoricity of T to get non-categoricity of T' via Ryll-Nardjewski, for a contradiction.

There is a relationship between categoricity and completeness:

Łos-Vaught Theorem. Let T be a countable theory with no finite models. If T is κ-categorical for some infinite κ, then T is complete.

Exercise: prove this, using Löwenheim-Skolem.

4. Elimination of Quantifiers

A theory T in a language $\mathcal{L}$ *admits elimination of quantifiers* (*has quantifier elimination* or some conglomerate) if for every $\mathcal{L}$-formula $\varphi(\bar{x})$ there is a quantifier-free formula $\psi(\bar{x})$ such that $T \models \varphi \leftrightarrow \psi$. Note the equivalence is *over* T.

Example 1. In Th($\mathbb{R}$), $\varphi(a, b, c) := \exists x(ax^2 + bx + c = 0)$ is equivalent, by the quadratic formula, to the quantifier free formula

$$\left[\left(a \neq 0 \;\&\; b^2 - 4ac > 0\right) \vee (a = 0 \;\&\; (c = 0 \;\vee\; b \neq 0))\right].$$

This is not true in Th($\mathbb{Q}$).
Example 2. Th($\mathbb{Q}$) (as a dense linear order) admits elimination of quantifiers. In fact we can semi-explicitly construct a quantifier-free equivalent for any given formula over Th($\mathbb{Q}$). If the original formula has free variables $x_1, \ldots, x_n$, the quantifier-free version will be a disjunction of conjunctions of atomic formulas of the form $x_i = x_j$ and $x_i < x_j$ for $i, j \leq n$.
Example 3. RCF does not admit quantifier elimination. In RCF, the $<$ ordering is definable, but requires quantifiers. This is actually the whole problem, in the sense that if we put RCF in the language of *ordered* fields, it *does* admit elimination of quantifiers.
Example 4. More theories with quantifier elimination: vector spaces over a field K, atomless Boolean algebras, algebraically closed fields. More theories without quantifier elimination: number theory, ZFC.

Why do this? Quantifier-free formulas are easier. For example, in $(\mathbb{N}, +, \cdot, <)$, the quantifier-free formulas are polynomial equations and inequalities. However, if we allow a single existential quantifier, we can define all c.e. sets. Quantifier elimination has connections to embeddings, decidability proofs, and structure classification.

If $\mathcal{L}$ is a recursive language (that is, there is a recursive coding of $\mathcal{L}$ that allows operations like forming conjunctions and substituting terms for free variables to be recursive), the *decision problem* for an $\mathcal{L}$-theory T is finding an algorithm to determine whether $T \models \varphi$ for $\mathcal{L}$-sentences φ, or showing there is no such algorithm. If the r.e. theory T has quantifier elimination and the elimination function is recursive, then T is decidable, by cobbling together the elimination functions with the function that tells us whether each quantifier-free sentence is provable or refutable form T (by finding a proof or a refutation). In fact, most of the languages we care about are recursive, and most theories that admit elimination of quantifiers do so in a way that gives a recursive elimination function.

A theory T is *model complete* if whenever $\mathcal{M} \subseteq \mathcal{N}$ are both models of T, $\mathcal{M} \preceq \mathcal{N}$ (the inclusion is an elementary embedding). Equivalently, every embedding between models of T is elementary. Model completeness for T is equivalent to all $\mathcal{L}$-formulas being equivalent over T to universal formulas, and so is implied by elimination of quantifiers.

Quantifier elimination allows a simple description of all the complete extensions of a theory, by reducing them to just the quantifier-free formulas. Because of this, it eases the classification of models up to elementary equivalence.

Elimination of quantifiers is a place where we see the interaction between syntax and semantics, or in other words between formulas and models. If T has elimination of quantifiers and $\mathcal{M}, \mathcal{N}, \mathcal{B}$ are substructures of models of T such that $\mathcal{M}$ embeds in $\mathcal{N}$ by f and in $\mathcal{B}$ by g, where these embeddings are not necessarily elementary, then there is a substructure $\mathcal{D}$ of a model of T such that $\mathcal{N}$ and $\mathcal{B}$ embed into $\mathcal{D}$ via some maps h_1, h_2, respectively, and the following diagram commutes.

$$\begin{tikzcd} & \mathcal{N} \arrow[dr, "h_1"] & \\ \mathcal{M} \arrow[ur, "f"] \arrow[dr, "g"'] & & \mathcal{D} \\ & \mathcal{B} \arrow[ur, "h_2"'] & \end{tikzcd}$$

Less purely model-theoretic is the following:

Suppose for all quantifier-free $\varphi(\bar{v}, w)$ and all $\mathcal{M}, \mathcal{N} \models T$ sharing some common substructure $\mathcal{A}$, if $\bar{a} \in A$ and $b \in M$ are such that $\mathcal{M} \models \varphi(\bar{a}, b)$, there is some $c \in N$ such that $\mathcal{N} \models \varphi(\bar{a}, c)$. Then T admits elimination of quantifiers.

5. Skolemization

Skolem's goal was to prove we don't need uncountability (he proved the existence of a countable model of ZFC). His method was to add functions to a language $\mathcal{L}$, based on a theory T, such that any substructure of a model $\mathcal{M}$ of T that is closed under those functions is an elementary substructure of $\mathcal{M}$. Then, prove a countable such substructure exists. These functions are now called *Skolem functions* and have other uses than proving existence of countable models.

Definition. A *Skolem function* for an $\mathcal{L}$-formula $\varphi(v, \bar{w})$ is a function f_φ such that $T \models \forall \bar{w}((\exists v \varphi(v, \bar{w}) \rightarrow \varphi(f_\varphi(\bar{w}), \bar{w}))$. T has *built-in Skolem functions* if for every φ there is such an f_φ in T. We may also call such a T a *Skolem theory.*

As Hodges says, "in a state of nature there are very few Skolem theories." However, they live "over" all our ordinary theories.

Definition. An *expansion* of a model $\mathcal{M} \models T$ to a model $\mathcal{M}' \models T'$, $T \subseteq T'$, adds no new elements and does not change interpretations of elements of T, but adds interpretations for the functions, relations, and constants of $T' - T$. If we ignore those interpretations in $\mathcal{M}'$ we get back to $\mathcal{M}$, and call it a *reduct* of $\mathcal{M}'$. A *conservative extension* of the $\mathcal{L}$-theory T is an $\mathcal{L}'$-theory T', $T \subseteq T'$ and $\mathcal{L} \subseteq \mathcal{L}'$, such that every $\mathcal{M} \models T$ has an expansion to a model $\mathcal{M}' \models T'$.

Theorem. Every theory T has a conservative extension T' with built-in Skolem functions. Further, we can choose the language $\mathcal{L}'$ of T' such that $|\mathcal{L}'| = \min\{|\mathcal{L}|, \aleph_0\}$.

We call T' a *Skolemization* of T, or sometimes the *(iterated) Skolem expansion.*

Proof. We build a sequence of languages $\mathcal{L} = \mathcal{L}_0 \subseteq \mathcal{L}_1 \subseteq \ldots$, and $\mathcal{L}_i$ theories T_i such that $T = T_0 \subseteq T_1 \subseteq \ldots$.

Given $\mathcal{L}_i$, let $\mathcal{L}_{i+1} = \mathcal{L}_i \cup \{f_\varphi : \varphi(v, w_1, \ldots, w_n) \text{ an } \mathcal{L}\text{-function}, n \geq 1\}$, where f_φ is an n-ary function symbol. For each $\mathcal{L}_i$-formula $\varphi(v, \bar{w})$, let Ψ_φ be the sentence $\forall \bar{w}((\exists v \varphi(v, \bar{w}) \rightarrow \varphi(f_\varphi(\bar{w}), \bar{w}))$, and let $T_{i+1} = T_i \cup \{\Psi_\varphi : \varphi \text{ an } \mathcal{L}_i\text{-formula}\}$.

Now suppose $\mathcal{M} \models T_i$. To interpret the function symbols of $\mathcal{L}_{i+1} - \mathcal{L}_i$, let c be some fixed element of $\mathcal{M}$. If $\varphi(v, \bar{w})$ is an $\mathcal{L}_i$-formula, with $\bar{w}$ an n-tuple, we can find a function $g : M^n \rightarrow M$ such that if $\bar{a} \in M^n$ and $X_{\bar{a}} := \{b \in M : \mathcal{M} \models \varphi(b, \bar{a})\}$ is nonempty, then $g(\bar{a}) \in X_{\bar{a}}$, and if $X_{\bar{a}} = \emptyset$, then $g(\bar{a}) = c$. We may then interpret f_φ as g and have $\mathcal{M} \models \Psi_\varphi$.

Let $\mathcal{L}' = \bigcup_i \mathcal{L}_i$ and $T' = \bigcup_i T_i$. If $\varphi(v, \bar{w})$ is an $\mathcal{L}'$-formula, it is an $\mathcal{L}_i$-formula for some i, and hence Ψ_φ is a sentence of T_{i+1} and T' has built-in Skolem functions. Iteration of the model expansion shows for any $\mathcal{M} \models T$ was can interpret the symbols of $\mathcal{L}' - \mathcal{L}$ to make $\mathcal{M} \models T'$. $\square$

The Skolemization of T is model-complete, as defined in §4. Existential quantifiers may be replaced by Skolem functions, making every $\mathcal{L}'$-formula T'-equivalent to a universal formula.

By adding something slightly stronger than Skolem functions we can get the following:

Theorem. Every theory T has a conservative extension T' which admits elimination of quantifiers.

Proof (sketch). For each formula $\varphi(\bar{x})$ of $\mathcal{L}$, add a new relation symbol (a *Skolem relation*) $R_\varphi(\bar{x})$ to $\mathcal{L}'$ and the axiom $\forall \bar{x}(\varphi(\bar{x}) \leftrightarrow R_\varphi(\bar{x}))$ to T'. Show that models can be expanded to interpret the Skolem relations. Finally, take any $\varphi'(\bar{x})$ in $\mathcal{L}'$, replace the Skolem relations with the formulas they match to get a formula $\varphi(\bar{x})$ of $\mathcal{L}$, and note that it is T'-equivalent to $R_\varphi(\bar{x})$ in $\mathcal{L}'$, so T' admits elimination of quantifiers. $\square$