Introduction to Model Theory
Some of the Formal Definitions

language: $\mathcal{L}$, a set of symbols for functions with arity, relations with arity, and constants. For example: $\{+, \cdot, \leq, =, 0, 1\}$, where the first four are binary functions and relations as expected and the last two are constants; $\{f, g, h, R, S, T, a, b, c\}$ where f and R are unary, g and h are binary, S is 4-ary and T is 5-ary, with a, b, and c constants. Equality is often assumed, and that is the convention we will follow.

$\mathcal{L}$-formulas: These are built from smaller pieces in the way you'd expect, using $\mathcal{L}$, a countably infinite set of variables, and standard logical symbols. First, *terms* are constants, variables, and functions of $\mathcal{L}$ applied to terms (such as $x+y$, though this is informally written). *Atomic $\mathcal{L}$-formulas* are equalities of terms (such as $x+y = x \cdot z$) and relations of $\mathcal{L}$ applied to terms (such as $x + y \leq 0$). Finally, the *$\mathcal{L}$-formulas* are the closure of the atomic $\mathcal{L}$-formulas under negation, conjunction, disjunction, and quantification (so if φ is a formula and v a variable, $\exists v\varphi$ and $\forall v\varphi$ are formulas). There should be more careful parentheses use than what I have written here. A *sentence* is a formula where every variable is bound by a quantifier; none are free.

$\mathcal{L}$-structure: $\mathcal{M}$, consisting of a set M (the universe, over which the quantifiers of $\mathcal{L}$-formulas will range) and *interpretations* of the functions, relations, and constants of $\mathcal{L}$, denoted by putting a superscript $\mathcal{M}$ on the original symbol. For example, if $\mathcal{L} = \{a, i, e\}$ consists of a binary function, unary function, and constant, one possible $\mathcal{L}$-structure is a group, where a is interpreted as the group addition, i as the function taking an element to its additive inverse, and e as the additive identity. As an $\mathcal{L}$-structure, we might write that group as $\mathcal{G} = (G, a^{\mathcal{G}}, i^{\mathcal{G}}, e^{\mathcal{G}})$. The cardinality of a structure $\mathcal{M}$ is $|M|$.

theories and models: An $\mathcal{L}$-theory is a (possibly infinite) set of $\mathcal{L}$-sentences, often written T or Σ. It is common for "theory" to mean "set of sentences closed under logical deduction", but we will not use it that way (nor does Marker). An $\mathcal{L}$-structure $\mathcal{M}$ is a *model* of a theory T, $\mathcal{M} \models T$, if every sentence of T is true in $\mathcal{M}$ (when interpreted as directed). For example, if we take $\mathcal{L} = \{a, i, e\}$ as above and let T be the axioms of an abelian group, the set of invertible 4×4 matrices with standard multiplication, multiplicative inverse, and identity matrix is *not* a model of T even though it is a perfectly good $\mathcal{L}$-structure. However, $\mathbb{Z}$ with the standard interpretations (addition, negation, 0) does model T.

syntax and semantics: We say $T \models \varphi$ for a theory T and $\mathcal{L}$-sentence φ if whenever $\mathcal{M} \models T$, $\mathcal{M} \models \varphi$ as well. We write $T \vdash \varphi$ if there is a proof of φ from T.[1] Gödel's completeness theorem says $T \models \varphi \;\leftrightarrow\; T \vdash \varphi$. This holds in particular for $\varphi = (\psi \;\&\; \neg\psi)$ — we can show a theory is consistent by showing it has a model (*is satisfiable*).

notes on satisfaction: *Compactness* says that a theory T is satisfiable if and only if every finite subset of T is satisfiable. Given T consistent and a sentence φ in the same language, at least one of $T \cup \{\varphi\}$ and $T \cup \{\neg\varphi\}$ is satisfiable. If for every φ exactly one of those holds, T is *complete*; when a complete theory is closed under $\models$,

[1] Warning: this is how I learned it and how Marker does it, but *not* how Hodges uses these symbols.

every $\mathcal{L}$-sentence or its negation is in the closure. Alternatively (but equivalently), the *theory of a structure* (or model) $\mathrm{Th}(\mathcal{M})$ is the set of sentences true in $\mathcal{M}$; a theory is complete if it is equivalent to $\mathrm{Th}(\mathcal{M})$ for some structure $\mathcal{M}$. Two theories are equivalent if they have the same logical consequences, or closure under logical deduction.

types: A type is a set of $\mathcal{L}$-formulas (not sentences) all involving the same finite sequence of free variables $\bar{v}$. If Γ is a type and $\mathcal{M}$ a structure, then $\mathcal{M}$ *realizes* Γ if there is some sequence $\bar{a}$ of elements of M such that for every $\varphi(\bar{v}) \in \Gamma$, $\mathcal{M} \models \varphi(\bar{a})$. Otherwise $\mathcal{M}$ *omits* Γ. Γ is *principal* with respect to a theory T if there is an $\mathcal{L}$-formula $\varphi(\bar{v})$ (a *generator* for Γ) such that $\exists\bar{v}\varphi(\bar{v})$ is consistent with T and for all $\gamma(\bar{v}) \in \Gamma$, $T \vdash (\varphi(\bar{v}) \to \gamma(\bar{v}))$.
Note: some authors (e.g., Marker) restrict types to being relative to some complete theory T. A type is a set of formulas *consistent with* T in that case.

Relationships between structures:

An $\mathcal{L}$-*embedding* from $\mathcal{M}$ to $\mathcal{N}$ is a one-to-one map that preserves the interpretation of all the symbols of $\mathcal{L}$. If it is bijective it is an $\mathcal{L}$-*isomorphism* and we write $\mathcal{M} \cong \mathcal{N}$. If $M \subseteq N$ and the inclusion map is an $\mathcal{L}$-embedding, we say $\mathcal{M}$ is a *substructure* of $\mathcal{N}$, which is an *extension* of $\mathcal{M}$.

Two $\mathcal{L}$-structures are *elementarily equivalent* if they model exactly the same sentences; this is written $\mathcal{M} \equiv \mathcal{N}$ (when not written $\mathrm{Th}(\mathcal{M}) = \mathrm{Th}(\mathcal{N})$). Isomorphic structures are always elementarily equivalent, but the reverse is not true.

An embedding $j : \mathcal{M} \to \mathcal{N}$ may be elementary too; this requires $\mathcal{M} \models \varphi(\bar{a}) \iff \mathcal{N} \models \varphi(j(\bar{a}))$ for all $\bar{a} \subseteq M$ and $\mathcal{L}$-formulas (not sentences) $\varphi(\bar{v})$. If this embedding is via inclusion, we say $\mathcal{M}$ is an elementary substructure and $\mathcal{N}$ an elementary extension, and write $\mathcal{M} \prec \mathcal{N}$.

A few exercises:

1. Prove that for any theory T and sentence φ, at least one of $T \cup \{\varphi\}$ or $T \cup \{\neg\varphi\}$ is consistent.
2. Prove compactness. That is, given an infinite theory T, show T is satisfiable if and only if every finite subset of T is satisfiable. You may assume completeness.
3. Let $\mathcal{L} = \emptyset$. Write an $\mathcal{L}$-sentence φ that is true in exactly those $\mathcal{L}$-structures with at least three elements, and a sentence ψ true in exactly those structures with exactly three elements. Finally, describe an infinite set of $\mathcal{L}$-sentences T such that $\mathcal{M} \models T$ if and only if M is infinite.
4. Show that if a theory has arbitrarily large finite models, it must have an infinite model.
5. For $\mathcal{L}$ a finite language and $\mathcal{M}$ a finite $\mathcal{L}$-structure, show there is an $\mathcal{L}$-sentence φ such that $\mathcal{N} \models \varphi$ if and only if $\mathcal{N} \cong \mathcal{M}$.

Some Initial Questions

Given a language $\mathcal{L}$ and theory T, we might ask:

(1) What cardinalities are the models of T? Can we create theories with particular answers to this question? In general, does having a model of cardinality κ imply having one of any other cardinality λ?

(2) How many models (up to isomorphism) does T have in any given cardinality κ? If only one, we say it is κ-categorical. Does being κ-categorical imply being λ-categorical for any other cardinality λ?
(3) What relationships do the models of T have to each other? For example, is there a single model of T elementarily embeddable every other model of T? We call such a structure a prime model.
(4) What types can be realized? For example, an atomic model realizes only principal types, while a saturated model realizes essentially as many types as possible. Given a model $\mathcal{M}$ and an omitted type Γ consistent with $\mathrm{Th}(\mathcal{M})$, can we extend $\mathcal{M}$ elementarily to a model that realizes Γ?
(5) When are partial isomorphisms between models extendible? For example, if you know two models share a common elementary submodel, what does that allow you to say? In a homogeneous model, partial isomorphisms are highly extendible.

Suggested References:

David Marker, *Model Theory: An Introduction*. Springer, 2002. Well-organized and readable. Spends a lot more time on indiscernibles, stability, and strongly minimal sets than Hodges, and gives more algebraic examples.

Wilfrid Hodges, *A Shorter Model Theory*. Cambridge University Press, 1997. I like this book but it is idiosyncratic (not that Marker's is not, of course). It does include things Marker does not: amalgamation, more on saturation and categoricity, existentially closed structures. Gets its name from the fact that Hodges has a much larger book called *Model Theory* (Cambridge, 1993) of which this is an abridgement.

Jerome Keisler, Fundamentals of Model Theory. In *Handbook of Mathematical Logic*, Jon Barwise, ed., North Holland, 1977. This is a higher-level overview though still contains good details.

Chang and Keisler, *Model Theory*. North-Holland, 1990 (third edition). This is the tome, the book that has it all (along with big Hodges). Well, almost certainly it doesn't have it all, but at twice as long as the previous books and much more than twice as long as Keisler's article, it is a good place to look when you can't find it somewhere else.