Notes on Definability

We may speak of a collection of structures as being ($\mathcal{L}$-)definable, meaning there is some sentence φ such that $\mathcal{M}$ is in the collection if and only if $\mathcal{M} \models \varphi$. If we want to extend to a set of sentences, such as $\mathrm{Th}(\mathcal{M})$ for some $\mathcal{M}$, we say the collection is $\mathcal{L}$-axiomatizable.

However, our primary use of the term is in definable subsets of a structure's universe. As we saw in seminar, the definable sets are a recursively defined class that starts with the set of all n-tuples for every n, tuples wherein some fixed pair of entries must be equal, graphs of functions, and relations. It is closed under complement, projection, Cartesian product, union, and intersection, and this gives you every set defined by a first-order formula.

One key fact is that defining specific sets typically relies on knowledge external to the model, such as using Lagrange's result that every positive integer is the sum of four squares to get the relation $x < y$ from the ring of integers, with no order relation built in. More broadly, a straightforward definition may mean different sets in different models. Consider the successor relation: given that $<$ is an irreflexive ordering, let $\varphi(x, y)$ mean $\forall z(x < y \ \& \ \neg(x < z \ \& \ z < y))$. In $\mathbb{N}$, this defines the set of pairs $(n, n+1)$. In $\mathbb{Q}$, however, it defines the empty set. If we venture outside total orders, this might define sets wherein more than one pair may have the same first or second entry. For example, in the complete binary tree $2^{<\omega}$, the set defined by the successor formula will contain $(\sigma, \sigma^\frown i)$ for all finite binary strings σ (including the empty string) and $i = 0, 1$ (where $\frown$ indicates concatenation).

A *minimal* structure $\mathcal{M}$ is an infinite structure in which the only first-order definable subsets (even with parameters) are finite or cofinite. We may also say an infinite definable set $X \subseteq M$ is minimal if for all definable $Z \subseteq M$, either $X \cap Z$ or $X - Z$ is finite.

Showing structures are minimal usually requires elimination of quantifiers, which is the statement that over the theory T, every first-order sentence is equivalent to a quantifier-free sentence. T in this case would be the theory of the structure. Two examples without proof:

- In an algebraically closed field $\mathcal{F}$, every formula $\varphi(x)$ is equivalent to a Boolean combination of polynomial equations $p(x) = 0$ with coefficients from F. The solution set will be F or a finite subset.
- In an infinite vector space $\mathcal{V}$ over a field K, every formula $\varphi(x)$ is equivalent to a Boolean combination of linear equations $rx = a$ where $r \in K$ and $a \in V$. The set of vectors satisfying this will be V, a singleton, or $\emptyset$.

Note: if K is finite and $\mathcal{V}$ is infinite-dimensional, elimination of quantifiers is not necessary and we can show directly $\mathcal{V}$ is minimal. Furthermore, without parameters, the only definable subsets are $\emptyset$, $\{0\}$, $V - \{0\}$, and V itself. To sketch the proof: the language of a vector space is the constant 0, the binary function $+$ interpreted as vector addition, and unary functions k for every $k \in K$, representing scalar multiplication by k. Without parameters you can only access statements of the form "$\boldsymbol{v} = 0$" or "$k\boldsymbol{v} = 0$"; via negations, conjunctions, and disjunctions you can get all four sets, but only those four. With parameters, you are naming some finite set of vectors; the span of that set, its subsets, and their complements are definable. This is where it is key that K is finite and $\mathcal{V}$ is infinite-dimensional: we can't get a basis for $\mathcal{V}$ in finitely-many parameters, and the span of any finite collection is finite.

Now, some commentary on definability in the contexts of computability theory and set theory. In computability, $\mathcal{E} = (\{W \subseteq \omega : W \text{ c.e.}\}, \subseteq)$ is a central structure. This is a partial order in which least upper bound and greatest lower bound always exist. As an exercise, write formulas that define LUB and GLB (here, union and intersection), formulas defining least and greatest element (here, $\emptyset$ and ω), and a formula defining complement pairs for any relation that is a partial order.[1] Those formulas will have three free variables in the first two cases, one free variable in the second two, and two in the last. Given those definitions, we can define the finite sets inside $\mathcal{E}$ by taking advantage of another easy fact from computability: a set is computable if and only if both it and its complement are computably enumerable – so in $\mathcal{E}$ as a poset, they are exactly the complemented elements. The finite sets are the only sets with solely computable sets as subsets, by another straightforward computability exercise that every infinite computable set has an infinite noncomputable c.e. subset. Therefore we may define them via the formula $\varphi(X)$:

$$\forall Y \exists Z\, (Y \subseteq X \;\rightarrow\; (Y \cap Z = \emptyset \;\&\; Y \cup Z = \omega)).$$

Working up to finite difference is a key tool in computability: it allows priority arguments with injury to be possible. The definability of the finite sets in $\mathcal{E}$ is part of a vital "interchangeability" of sets that only differ by finitely many elements (for virtually everything we want to do, being wrong on a finite initial segment is "close enough") that one could argue is the whole reason computability theory can exist as its own discipline (somewhat analogous to the completeness theorem in model theory).

Finally, set theory. On a very basic level, the language of set theory has a single binary predicate, $\in$, and we rely on the definability (in ZFC; that is, in every model of ZFC) of important concepts from subset to ordinal. The application of the Incompleteness Theorem to ZFC comes from the fact that in ZFC we can define a model of Peano Arithmetic.

Forcing, which extends a model of ZFC to a larger model with the same ordinals, can change many things about X (for example, whether X is countable), but it does not change whether a relation is a well-ordering. This is because both being a well-ordering and not being a well-ordering can be defined existentially. (You are not a well-ordering if you have an infinite descending chain; you are a well-ordering if you have a function assigning every element an ordinal rank.) The witness to well-ordering or ill-ordering in a model is still a witness in an extension of that model.[2]

The Axiom of Choice implies that the reals can be well-ordered, but it says nothing about the definability of such a well-ordering. The Axiom of Determinacy implies that the reals cannot be well-ordered; thus, it contradicts Choice. Fragments of Determinacy, which postulate determinacy for sets of reals that are definable at a certain level, can, however, be consistent with Choice. They begin with Borel Determinacy (determinacy for Borel, or Δ^1_1-definable, sets), which is out and out provable in ZFC, and go up to fragments of higher and higher consistency strength. Generally, Determinacy at a certain level of definability implies that no well-ordering of the reals can be defined at that level.

The proof of the consistency of the Axiom of Choice is as follows: Inside any model of ZFC you can define the "constructible universe" $L = \bigcup_\alpha L_\alpha$. $L_0 = \emptyset$; $L_{\alpha+1}$ consists of all the definable subsets of the structure $(L_\alpha, \in)$; and for λ a limit ordinal, L_λ is the union of the L_α for $\alpha < \lambda$. In the structure $(L, \in)$, not only does the Axiom of Choice hold, but there is a definable well-ordering of the universe.

[1] Notice that these formulas are valid for any binary relation; they may or may not have meaning useful to us as mathematicians, but they still define sets.

[2] Anything with a universal quantifier is open to change, since the domain of the quantifier has increased.