Completeness Compactness and an application to Ramsey's Theorem

Dartmouth Logic Seminar April 6, 2009 Marcia J. Groszek

None of the results in this talk are due to me.

A first-order *formal language* for talking about a kind of mathematical structure (for example, the language of ordered rings) has

formulas, such as (x < y + ((1 + 1) \* x));

sentences, such as  $(\forall x)(x + 0 = x)$ , which are formulas with no free variables;

built from:

logical symbols:  $\forall$ ,  $\exists$ , (, ),  $\neg$  (not),  $\land$  (and),  $\lor$  (or),  $\rightarrow$ ,  $\leftrightarrow$ , =, x, y, z...

(parameter) constant (0, 1), function (+, \*), and predicate (<) symbols.

A structure for the language consists of a nonempty set X (the *universe*), together with elements of X, functions on X, and relations on X designated by the constant, function, and relation symbols of the language.

Note that not every structure for the language of ordered rings is an ordered ring.

A structure is a *model* for a set  $\Sigma$  of sentences (sometimes called axioms) if all the sentences of  $\Sigma$  are true in that model.

An ordered ring is a structure for this language that is a model of the axioms for an ordered ring. A set of axioms  $\Sigma$  logically implies a sentence  $\alpha$ ,

 $\Sigma \models \alpha,$ 

if  $\alpha$  is true in every model of  $\Sigma$ .

So, if  $\Sigma$  is the set of axioms for an ordered ring,  $\Sigma \models \alpha$  just in case  $\alpha$  is true in every ordered ring.

 $\{\alpha \mid \Sigma \models \alpha\}$  is the theory with axioms  $\Sigma$ ; in this example, it is the theory of ordered rings, in other words, the set of sentences true in every ordered ring.

A deduction (formal proof) of  $\alpha$  from  $\Sigma$  is a finite sequence of formulas, each one of which either is in  $\Sigma$ , or is in a certain set  $\Lambda$  of logical axioms, or is derived from earlier formulas in the sequence using one of a given set of logical rules.

Example logical axiom:  $((\forall x(x + 0 = x)) \rightarrow (1 + 0 = 1)).$ 

Example logical rule: From  $\alpha$  and  $(\alpha \rightarrow \beta)$ , derive  $\beta$ .

If there is a deduction of  $\alpha$  from  $\Sigma$ , we say  $\alpha$  is deducible from  $\Sigma$ ,

$$\Sigma \vdash \alpha$$
.

Key features of formal deductions:

Formal deductions are logically valid:

$$\Sigma \vdash \alpha \implies \Sigma \models \alpha.$$

If  $\Sigma$  is a finite set of axioms, then there is an algorithm that can tell whether a given finite sequence of formulas is a deduction from  $\Sigma$  or not:

The set of deductions from  $\Sigma$  is effectively decidable, or computable.

IMPORTANT WARNING:

 $\{\alpha \mid \mathbf{\Sigma} \vdash \alpha\}$ 

is in general NOT decidable.

Although we can decide whether a given finite sequence of formulas is a deduction from  $\Sigma$ , to tell whether  $\Sigma \vdash \alpha$  we would have to examine infinitely many potential deductions.

In fact, in the language of ordered rings,

$$\{\alpha \mid \emptyset \vdash \alpha\}$$

is not decidable.

Gödel's Completeness Theorem:

## $\Sigma \models \alpha \iff \Sigma \vdash \alpha.$

For example, any sentence in the language of ordered rings is either provable from the axioms of ordered rings, or false in some ordered ring. Compactness Theorem:

If  $\Sigma$  is a set of sentences such that every finite subset of  $\Sigma$  has a model, then  $\Sigma$  itself has a model.

Proof: Suppose  $\Sigma$  has no model. Then, vacuously,  $\Sigma \models \alpha$  for every sentence  $\alpha$ . By the Completeness Theorem, then,  $\Sigma \vdash \alpha$  for every sentence  $\alpha$ . For example,

 $\Sigma \vdash (\exists x)(x \neq x).$ 

Because deductions are finite, there is a finite subset  $\Delta \subseteq \Sigma$  such that

$$\Delta \vdash (\exists x)(x \neq x).$$

But then  $\Delta$  is a finite subset of  $\Sigma$  with no model.

We use the Compactness Theorem to prove the finitary version of Ramsey's Theorem from the infinitary version.

First, some notation:

If X is any set, and  $n \in \omega$  ( $\omega = \mathbb{N}$ ),

$$[X]^n = \{Y \subseteq X \mid |Y| = n\}.$$

A coloring of X in k colors is a function

 $c: X^n \to P$ 

where P is some set of size k.

A subset  $H \subset X$  is homogeneous for c. or monochromatic, if for some color i,

$$(\forall Y \in [H]^n)(c(Y) = i).$$

If a and b are cardinal numbers (natural numbers or  $\omega$  for our purposes), and n and k are natural numbers, then

$$a \to (b)_k^n$$

means that if A is a set of size a, for every coloring of  $[A]^n$  in k colors, there is a homogenous subset  $H \subset A$  of size b.

Ramsey's Theorem (Infinitary Version):

$$(\forall n \in \omega)(\forall k \in \omega)(\omega \to (\omega)_k^n).$$

Ramsey's Theorem (Finitary Version):

 $(\forall n \in \omega)(\forall k \in \omega)(\forall b \in \omega)(\exists a \in \omega)(a \to (b)_k^n).$ 

Proof of the finitary version of Ramsey's Theorem from the infinitary version:

Suppose the finitary version fails. Then, for some n, k, and b, for no  $a \in \omega$  do we have

$$(a \rightarrow (b)_k^n).$$

That is, for every  $a \in \omega$  it is possible to color *n*-element subsets of a size *a* set in *k* colors so that no size *b* subset is monochromatic.

For typographical ease, we assume n = k = 2.

This means the following set  $\Sigma$  of sentences has models of arbitrarily large finite size:

Our language has symbols R and B, for colors red and blue. We interpret the formula Rxyto mean that  $\{x, y\}$  is assigned color red. We include in our set  $\Sigma$  axioms saying that this is really a coloring of sets of size 2:

$$(\forall x)(\forall y)(x = y \to (\neg Rxy \land \neg Bxy));$$
  
 $(\forall x)(\forall y)(x \neq y \to (Rxy \leftrightarrow \neg Bxy));$   
 $(\forall x)(\forall y)(x \neq y \to (Rxy \leftrightarrow Ryx)).$ 

We also include a sentence saying there is no size b homgeneous set:

$$(\forall x_1) \cdots (\forall x_b) ((\bigwedge_{1 \le i < j \le b} x_i \ne x_j) \rightarrow (\bigvee_{1 \le i < j \le b} Rx_i x_j) \land (\bigvee_{1 \le i < j \le b} Bx_i x_j)).$$

A model for  $\Sigma$  is a set X with a coloring of  $[X]^2$  in colors R and B having no homogenous set of size b.

For every  $a \in \omega$ , because we do not have

$$(a \rightarrow (b)_2^2),$$

we do have a set X of size a with a coloring of  $[X]^2$  in colors R and B having no homogeneous set of size b. That is,  $\Sigma$  has a model of size a, for every finite a.

Let  $\sigma_a$  be a sentence saying there are at least a elements in the universe:

$$(\exists x_1) \cdots (\exists x_a) (\bigwedge_{1 \le i < j \le a} x_i \ne x_j).$$

Because  $\Sigma$  has arbitrarily large finite models, every finite subset of  $\Sigma'$  has a model, where

$$\Sigma' = \Sigma \cup \{\sigma_a \mid a \in \omega\}.$$

By Compactness,  $\Sigma'$  has a model, where

$$\Sigma' = \Sigma \cup \{\sigma_a \mid a \in \omega\}.$$

That is, there is an infinite set X with a coloring of  $[X]^2$  in two colors with no homogeneous set of size b.

By restricting to a subset of X (if necessary), we can assume X is countable,  $|X| = \omega$ .

If there is no homogenous set of size b, certainly there is no homogeneous set of size  $\omega$ . That is, we have shown

$$\omega \not\rightarrow (\omega)_2^2.$$

Hence the finitary version of Ramsey's Theorem (for n = k = 2) follows from the infinitary version, via Compactness. Here is a proof of the infinitary version of Ramsey's Theorem (for n = k = 2). It's much easier than the proof for the finitary case (but gives less combinatorial information).

R B R B B R R 8 RB7 B B R B B RB R B B R B6 5 B R R B R 4  $R \quad B \quad R \quad R$ 3  $R \quad B \quad R$ 2 R R1 B0 1 2 3 4 5 6 7 0

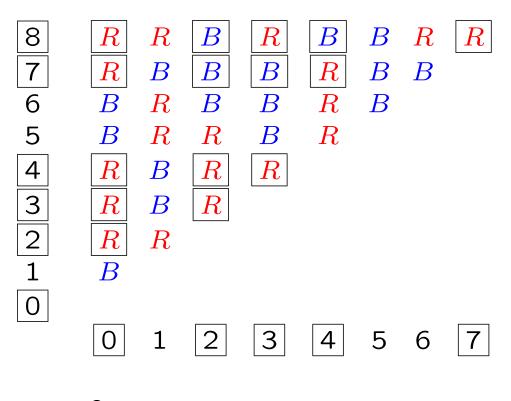
A coloring of  $[\omega]^2$  in two colors.

The pair (a, b), where a < b, represents the set  $\{a, b\}$ .

R R R R R R R R R 81 56 R RR RR RR55 *R R* R R R R R $34 \quad R \quad R \quad R \quad R \quad R \quad R$  $30 \quad R \quad R \quad R \quad R$  $24 \quad R \quad R \quad R$  $20 \quad R \quad R$ 7 R0 7 20 24 30 34 55 56 0

 $H = \{0, 7, 20, 24, 30, 34, 55, 56, 81, ...\}$  is homogeneous in color red.

To construct a homogeneous set:



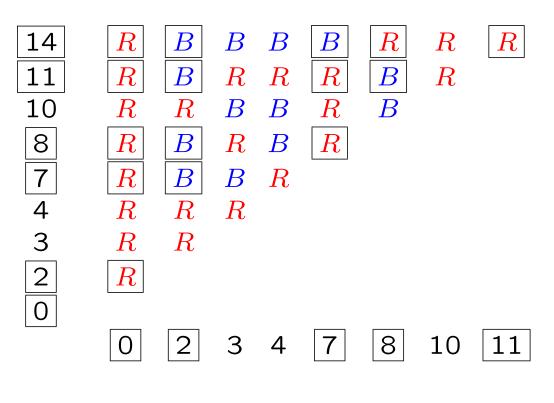
 $x_0 = 0$ 

 $X_0 = \{b \in \omega - \{x_0\} \mid c(\{0, b\}) = R\}$  if this is infinite;

 $X_0 = \{b \in \omega - \{x_0\} \mid c(\{0, b\}) = B\}$  otherwise.

 $C(0) = \mathbb{R}$  or  $C(0) = \mathbb{B}$ , respectively.

## RBBB B R14 R R11 В RRRRBR10 RRBBRB8 RBRBR7 RBBR4 RRR3 RR2 R0 2 3 4 7 8 10 11

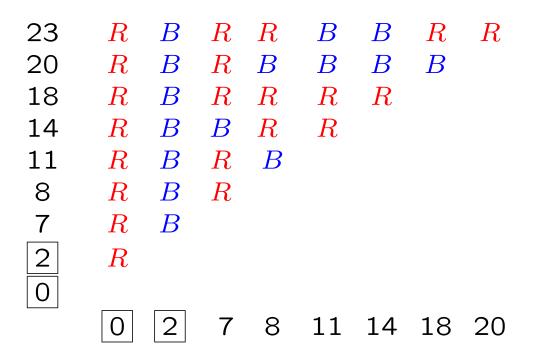


 $x_1 = \min(X_0)$ 

 $X_1 = \{b \in X_0 - \{x_1\} \mid c(\{x_1, b\}) = R\}$  if this is infinite;

 $X_1 = \{b \in X_0 - \{x_1\} \mid c(\{x_1, b\}) = B\}$  otherwise.

 $C(x_1) = \mathbb{R}$  or  $C(x_1) = \mathbb{B}$ , respectively.



Continue:  $x_{n+1} = min(X_n)$ 

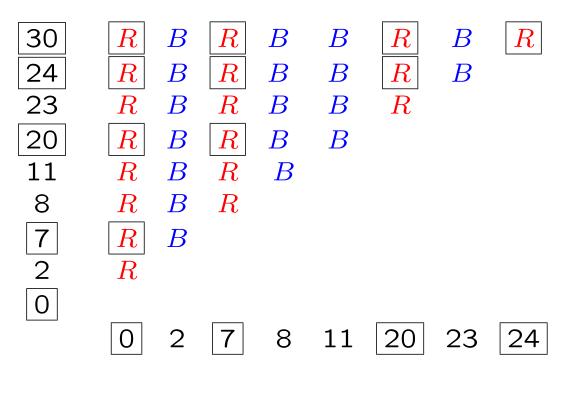
 $X_{n+1} = \{b \in X_n - \{x_{n+1}\} \mid c(\{x_{n+1}, b\}) = R\}$  if this is infinite;

 $X_{n+1} = \{b \in X_n - \{x_{n+1}\} \mid c(\{x_{n+1}, b\}) = B\}$ otherwise.

 $C(x_{n+1}) = \mathbb{R}$  or  $C(x_{n+1}) = \mathbb{B}$ , respectively.

## 30 RB R B B RB R24 RRBB BRBB R B B23 RR20 *R B R B* BR B R B 11 8 *R B R* 7 *R B* 2 R0 2 7 8 11 20 23 24 0

 $X = \{x_0, x_1, x_2, \ldots, x_n, \ldots\}.$ 



 $H = \{x \in X \mid C(x) = \mathbb{R}\}$  if this is infinite;

 $H = \{x \in X \mid C(x) = B\}$  otherwise.

## 81 RRRRRRRR56 RRRRRRRR R55 R R $R \quad R$ 34 RR $R \quad R$ R30 RR $R \quad R$ 24 R RR20 R R7 R0 7 20 24 30 34 55 56 0

H is homogeneous in color R or B, respectively.