Completeness
Compactness
and an application to
Ramsey's Theorem

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None of the results in this talk are due to me.

A first-order *formal language* for talking about a kind of mathematical structure (for example, the language of ordered rings) has

formulas, such as $(x < y + ((1 + 1) * x))$;

sentences, such as $(\forall x)(x + 0 = x)$, which are formulas with no free variables;

built from:

logical symbols: $\forall$, $\exists$, (,), $\neg$ (not), $\wedge$ (and), $\vee$ (or), $\rightarrow$, $\leftrightarrow$, $=$, x, y, $z \ldots$

(parameter) constant (0, 1), function ($+$, $*$), and predicate ($<$) symbols.

A *structure* for the language consists of a nonempty set X (the *universe*), together with elements of X, functions on X, and relations on X designated by the constant, function, and relation symbols of the language.

Note that not every structure for the language of ordered rings is an ordered ring.

A structure is a *model* for a set Σ of sentences (sometimes called axioms) if all the sentences of Σ are true in that model.

An ordered ring is a structure for this language that is a model of the axioms for an ordered ring.

A set of axioms Σ logically implies a sentence α,

$$\Sigma \models \alpha,$$

if α is true in every model of Σ.

So, if Σ is the set of axioms for an ordered ring, $\Sigma \models \alpha$ just in case α is true in every ordered ring.

$\{\alpha \mid \Sigma \models \alpha\}$ is the theory with axioms Σ; in this example, it is the theory of ordered rings, in other words, the set of sentences true in every ordered ring.

A *deduction* (formal proof) of α from Σ is a finite sequence of formulas, each one of which either is in Σ, or is in a certain set Λ of logical axioms, or is derived from earlier formulas in the sequence using one of a given set of logical rules.

Example logical axiom:
$((\forall x(x + 0 = x)) \rightarrow (1 + 0 = 1))$.

Example logical rule:
From α and $(\alpha \rightarrow \beta)$, derive β.

If there is a deduction of α from Σ, we say α is deducible from Σ,

$$\Sigma \vdash \alpha.$$

Key features of formal deductions:

Formal deductions are logically valid:

$$\Sigma \vdash \alpha \implies \Sigma \models \alpha.$$

If Σ is a finite set of axioms, then there is an algorithm that can tell whether a given finite sequence of formulas is a deduction from Σ or not:

The set of deductions from Σ is effectively decidable, or computable.

IMPORTANT WARNING:

$$\{\alpha \mid \Sigma \vdash \alpha\}$$

is in general NOT decidable.

Although we can decide whether a given finite sequence of formulas is a deduction from Σ, to tell whether $\Sigma \vdash \alpha$ we would have to examine infinitely many potential deductions.

In fact, in the language of ordered rings,

$$\{\alpha \mid \emptyset \vdash \alpha\}$$

is not decidable.

Gödel's Completeness Theorem:

$$\Sigma \models \alpha \iff \Sigma \vdash \alpha.$$

For example, any sentence in the language of ordered rings is either provable from the axioms of ordered rings, or false in some ordered ring.

Compactness Theorem:

If Σ is a set of sentences such that every finite subset of Σ has a model, then Σ itself has a model.

Proof: Suppose Σ has no model. Then, vacuously, $\Sigma \models \alpha$ for every sentence α. By the Completeness Theorem, then, $\Sigma \vdash \alpha$ for every sentence α. For example,

$$\Sigma \vdash (\exists x)(x \neq x).$$

Because deductions are finite, there is a finite subset $\Delta \subseteq \Sigma$ such that

$$\Delta \vdash (\exists x)(x \neq x).$$

But then Δ is a finite subset of Σ with no model.

We use the Compactness Theorem to prove the finitary version of Ramsey's Theorem from the infinitary version.

First, some notation:

If X is any set, and $n \in \omega$ ($\omega = \mathbb{N}$),

$$[X]^n = \{Y \subseteq X \mid |Y| = n\}.$$

A coloring of X in k colors is a function

$$c : X^n \to P$$

where P is some set of size k.

A subset $H \subset X$ is homogeneous for c. or monochromatic, if for some color i,

$$(\forall Y \in [H]^n)(c(Y) = i).$$

If a and b are cardinal numbers (natural numbers or ω for our purposes), and n and k are natural numbers, then

$$a \to (b)^n_k$$

means that if A is a set of size a, for every coloring of $[A]^n$ in k colors, there is a homogenous subset $H \subset A$ of size b.

Ramsey's Theorem (Infinitary Version):

$$(\forall n \in \omega)(\forall k \in \omega)(\omega \to (\omega)^n_k).$$

Ramsey's Theorem (Finitary Version):

$$(\forall n \in \omega)(\forall k \in \omega)(\forall b \in \omega)(\exists a \in \omega)(a \to (b)^n_k).$$

Proof of the finitary version of Ramsey's Theorem from the infinitary version:

Suppose the finitary version fails. Then, for some n, k, and b, for no $a \in \omega$ do we have

$$(a \to (b)^n_k).$$

That is, for every $a \in \omega$ it is possible to color n-element subsets of a size a set in k colors so that no size b subset is monochromatic.

For typographical ease, we assume $n = k = 2$.

This means the following set Σ of sentences has models of arbitrarily large finite size:

Our language has symbols R and B, for colors red and blue. We interpret the formula Rxy to mean that $\{x, y\}$ is assigned color red. We include in our set Σ axioms saying that this is really a coloring of sets of size 2:

$$(\forall x)(\forall y)(x = y \rightarrow (\neg Rxy \ \wedge \neg Bxy));$$

$$(\forall x)(\forall y)(x \neq y \rightarrow (Rxy \ \leftrightarrow \neg Bxy));$$

$$(\forall x)(\forall y)(x \neq y \rightarrow (Rxy \ \leftrightarrow \ Ryx)).$$

We also include a sentence saying there is no size b homgeneous set:

$$(\forall x_1) \cdots (\forall x_b)((\bigwedge_{1 \leq i < j \leq b} x_i \neq x_j) \rightarrow$$

$$\left(\bigvee_{1 \leq i < j \leq b} Rx_i x_j \right) \wedge \left(\bigvee_{1 \leq i < j \leq b} Bx_i x_j \right)).$$

A model for Σ is a set X with a coloring of $[X]^2$ in colors R and B having no homogenous set of size b.

For every $a \in \omega$, because we do not have

$$(a \rightarrow (b)^2_2),$$

we do have a set X of size a with a coloring of $[X]^2$ in colors R and B having no homogeneous set of size b. That is, Σ has a model of size a, for every finite a.

Let σ_a be a sentence saying there are at least a elements in the universe:

$$(\exists x_1) \cdots (\exists x_a)(\bigwedge_{1 \leq i < j \leq a} x_i \neq x_j).$$

Because Σ has arbitrarily large finite models, every finite subset of Σ' has a model, where

$$\Sigma' = \Sigma \cup \{\sigma_a \mid a \in \omega\}.$$

By Compactness, Σ' has a model, where

$$\Sigma' = \Sigma \cup \{\sigma_a \mid a \in \omega\}.$$

That is, there is an infinite set X with a coloring of $[X]^2$ in two colors with no homogeneous set of size b.

By restricting to a subset of X (if necessary), we can assume X is countable, $|X| = \omega$.

If there is no homogenous set of size b, certainly there is no homogeneous set of size ω. That is, we have shown

$$\omega \not\to (\omega)^2_2.$$

Hence the finitary version of Ramsey's Theorem (for $n = k = 2$) follows from the infinitary version, via Compactness.

Here is a proof of the infinitary version of Ramsey's Theorem (for $n = k = 2$). It's much easier than the proof for the finitary case (but gives less combinatorial information).

8	R	R	B	R	B	B	R	R
7	R	B	B	B	R	B	B	
6	B	R	B	B	R	B		
5	B	R	R	B	R			
4	R	B	R	R				
3	R	B	R					
2	R	R						
1	B							
0								
	0	1	2	3	4	5	6	7

A coloring of $[\omega]^2$ in two colors.

The pair (a, b), where $a < b$, represents the set $\{a, b\}$.

81	R	R	R	R	R	R	R	R
56	R	R	R	R	R	R	R	
55	R	R	R	R	R	R		
34	R	R	R	R	R			
30	R	R	R	R				
24	R	R	R					
20	R	R						
7	R							
0								
	0	7	20	24	30	34	55	56

$H = \{0, 7, 20, 24, 30, 34, 55, 56, 81, \ldots\}$ is homogeneous in color red.

To construct a homogeneous set:

$\boxed{8}$	$\boxed{R}$	R	$\boxed{B}$	$\boxed{R}$	$\boxed{B}$	B	R	$\boxed{R}$
$\boxed{7}$	$\boxed{R}$	B	$\boxed{B}$	$\boxed{B}$	$\boxed{R}$	B	B	
6	B	R	B	B	R	B		
5	B	R	R	B	R			
$\boxed{4}$	$\boxed{R}$	B	$\boxed{R}$	$\boxed{R}$				
$\boxed{3}$	$\boxed{R}$	B	$\boxed{R}$					
$\boxed{2}$	$\boxed{R}$	R						
1	B							
$\boxed{0}$								
	$\boxed{0}$	1	$\boxed{2}$	$\boxed{3}$	$\boxed{4}$	5	6	$\boxed{7}$

$x_0 = 0$

$X_0 = \{b \in \omega - \{x_0\} \mid c(\{0, b\}) = R\}$ if this is infinite;

$X_0 = \{b \in \omega - \{x_0\} \mid c(\{0, b\}) = B\}$ otherwise.

$C(0) = R$ or $C(0) = B$, respectively.

14	R	B	B	B	B	R	R	R
11	R	B	R	R	R	B	R	
10	R	R	B	B	R	B		
8	R	B	R	B	R			
7	R	B	B	R				
4	R	R	R					
3	R	R						
2	R							
$\boxed{0}$								
	$\boxed{0}$	2	3	4	7	8	10	11

$\boxed{14}$	$\boxed{R}$	$\boxed{B}$	B	B	$\boxed{B}$	$\boxed{R}$	R	$\boxed{R}$
$\boxed{11}$	$\boxed{R}$	$\boxed{B}$	R	R	$\boxed{R}$	$\boxed{B}$	R	
10	R	R	B	B	R	B		
$\boxed{8}$	$\boxed{R}$	$\boxed{B}$	R	B	$\boxed{R}$			
$\boxed{7}$	$\boxed{R}$	$\boxed{B}$	B	R				
4	R	R	R					
3	R	R						
$\boxed{2}$	$\boxed{R}$							
$\boxed{0}$								
	$\boxed{0}$	$\boxed{2}$	3	4	$\boxed{7}$	$\boxed{8}$	10	$\boxed{11}$

$x_1 = min(X_0)$

$X_1 = \{b \in X_0 - \{x_1\} \mid c(\{x_1, b\}) = R\}$ if this is infinite;

$X_1 = \{b \in X_0 - \{x_1\} \mid c(\{x_1, b\}) = B\}$ otherwise.

$C(x_1) = R$ or $C(x_1) = B$, respectively.

23	R	B	R	R	B	B	R	R
20	R	B	R	B	B	B	B	
18	R	B	R	R	R	R		
14	R	B	B	R	R			
11	R	B	R	B				
8	R	B	R					
7	R	B						
$\boxed{2}$	R							
$\boxed{0}$								
	$\boxed{0}$	$\boxed{2}$	7	8	11	14	18	20

Continue: $x_{n+1} = min(X_n)$

$X_{n+1} = \{b \in X_n - \{x_{n+1}\} \mid c(\{x_{n+1}, b\}) = R\}$ if this is infinite;

$X_{n+1} = \{b \in X_n - \{x_{n+1}\} \mid c(\{x_{n+1}, b\}) = B\}$ otherwise.

$C(x_{n+1}) = R$ or $C(x_{n+1}) = B$, respectively.

30	R	B	R	B	B	R	B	R
24	R	B	R	B	B	R	B	
23	R	B	R	B	B	R		
20	R	B	R	B	B			
11	R	B	R	B				
8	R	B	R					
7	R	B						
2	R							
0								
	0	2	7	8	11	20	23	24

$$X = \{x_0, x_1, x_2, \ldots, x_n, \ldots\}.$$

$\boxed{30}$	$\boxed{R}$	B	$\boxed{R}$	B	B	$\boxed{R}$	B	$\boxed{R}$
$\boxed{24}$	$\boxed{R}$	B	$\boxed{R}$	B	B	$\boxed{R}$	B	
23	R	B	R	B	B	R		
$\boxed{20}$	$\boxed{R}$	B	$\boxed{R}$	B	B			
11	R	B	R	B				
8	R	B	R					
$\boxed{7}$	$\boxed{R}$	B						
2	R							
$\boxed{0}$								
	$\boxed{0}$	2	$\boxed{7}$	8	11	$\boxed{20}$	23	$\boxed{24}$

$H = \{x \in X \mid C(x) = R\}$ if this is infinite;

$H = \{x \in X \mid C(x) = B\}$ otherwise.

81	R	R	R	R	R	R	R	R
56	R	R	R	R	R	R	R	
55	R	R	R	R	R	R		
34	R	R	R	R	R			
30	R	R	R	R				
24	R	R	R					
20	R	R						
7	R							
0								
	0	7	20	24	30	34	55	56

H is homogeneous in color R or B, respectively.