Low upper bound of ideals, coding into rich Π^0_1 classes

Antonín Kučera
the main part is a joint project with T. Slaman

Charles University, Prague

September 2007, Chicago

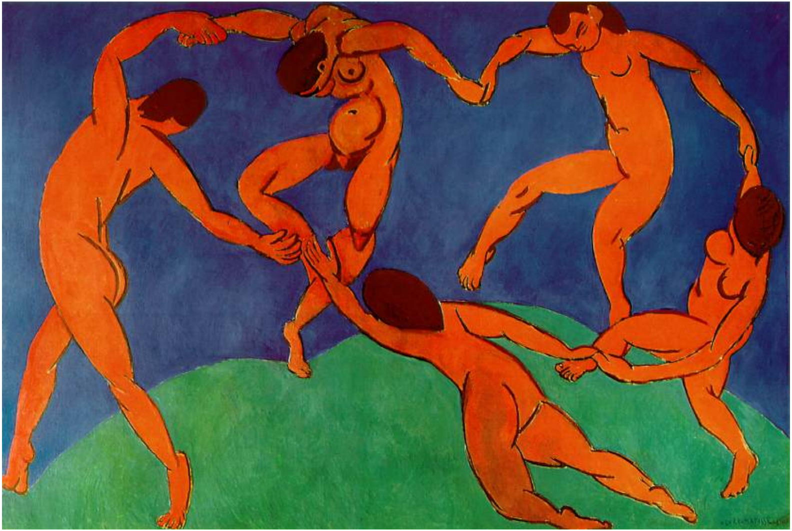
The main result

- There is a low T-upper bound for the class of K-trivials
- A characterization of ideals in Δ_2^0 degrees which have a low T-upper bound

Motto:

It is easy to handle ideals when they are low

Henri Matisse : The Dance



Algorithmic weakness

There are several notions of computational weakness related to 1-randomness

Definition

1. $\mathcal{L}$ denotes the class of sets which are low for 1-randomness, i.e. sets A such that every 1-random set is also 1-random relative to A.
2. $\mathcal{K}$ denotes the class of K-trivial sets, i.e. the class of sets A such that for all n, $K(A \upharpoonright n) \leq K(0^n) + O(1)$.
3. $\mathcal{M}$ denotes the class of sets that are low for K, i.e. sets A such that for all σ, $K(\sigma) \leq K^A(\sigma) + O(1)$.
4. A set A is a basis for 1-randomness if $A \leq_T Z$ for some Z such that Z is 1-random relative to A. The collection of such sets is denoted by $\mathcal{B}$.

Theorem (Nies, Hirschfeldt, Stephan)

$\mathcal{K} = \mathcal{L} = \mathcal{M} = \mathcal{B}$

More precisely:

- Nies: $\mathcal{L} = \mathcal{M}$
- Hirschfeldt, Nies: $\mathcal{K} = \mathcal{M}$
- Hirschfeldt, Nies, Stephan: $\mathcal{K} = \mathcal{B}$

Four different characterizations of the same class!
However, these characterizations yield different information content

Basic facts about $\mathcal{K}$

- $\mathcal{K} \subseteq \Delta^0_2$
- $\mathcal{K} \subseteq L_1$ (i.e. K-trivials are low)

More precisely:

- Chaitin: $\mathcal{K} \subseteq \Delta^0_2$
- A.K.: $\mathcal{L} \subseteq GL_1$ (thus, $\mathcal{L} = \mathcal{K} \subseteq L_1$)

Nowadays there are easier ways to prove lowness of K-trivials

Theorem (Nies; Downey, Hirschfeldt, Nies, Stephan)

- *r.e. K-trivial sets induce a Σ^0_3 ideal in the r.e. T-degrees*
- *K-trivial sets induce an ideal in the ω-r.e. T-degrees generated by its r.e. members (in fact, a Σ^0_3 ideal in the ω-r.e. T-degrees)*

Theorem (Downey, Hirschfeldt, Nies, Stephan; Nies)

- *There is an effective sequence $\{B_e, d_e\}_e$ of all the r.e. K-trivial sets and of constants such that each B_e is K-trivial via d_e*
- *There is no effective sequence $\{B_e, c_e\}_e$ of all the r.e. low for K sets with appropriate constants*
- *There is no effective way to obtain from a pair (B, d), where B is an r.e. set that is K-trivial via d, a constant c such that B is low for K via c*
- *There is no effective listing of all the r.e. K-trivial sets together with their low indices*

Theorem (Nies)

For each low r.e. set B, there is an r.e. K-trivial set A such that $A \not\leq_T B$.

Thus, no low r.e. set can be a T-upper bound for the class $\mathcal{K}$.

Comment

The proof uses Robinson low guessing technique which is compatible for r.e. sets with a technique do what is cheap. Cheap

is defined

- either by a cost function in case of K-trivials,
- or by having a small measure in case of low for random sets.

However, in the more general case of Δ^0_2 instead of r.e. sets, the Robinson low guessing technique does not seem to be compatible with a technique do what is cheap. In fact, it is not.

Since all K-trivials are low and every K-trivial set is recursive in some r.e. K-trivial set, we have, as a corollary, that the ideal (induced by) $\mathcal{K}$ is nonprincipal (in the Δ_2^0 T-degrees)

A more general result.

Theorem (Nies)

For any effective listing $\{B_e, z_e\}_e$ of low r.e. sets and of their low indices there is an r.e. K-trivial set A such that $A \not\leq_T B_e$ for all e.

This result is, in fact, used to prove that there is no effective way to obtain low indices of (r.e.) K-trivial sets

Theorem (Nies)

- *There is a low$_2$ r.e. set which is a T-upper bound for the class of K-trivials.*
- *Any proper Σ^0_3 ideal in the r.e. T-degrees has a low$_2$ r.e. T-upper bound*

Question

Is there a low Δ^0_2 T-upper bound for the class $\mathcal{K}$?

Theorem (Yates)

For any r.e. set A TFAE:

1. $A'' \equiv_T \emptyset''$
2. $\{x : W_x \leq_T A\}$ *is a* Σ^0_3 *set*
3. *the class* $\{W_x : W_x \leq_T A\}$ *is uniformly r.e.*

Together with Nies' result, we have the following characterization.

Fact

An ideal of r.e sets has a low_2 r.e. T-upper bound if and only if it is a subideal of a proper Σ^0_3 ideal.

Open

A characterization of Σ^0_3 ideals in the r.e. T-degrees for which there is a low T-upper bound, not necessarily r.e.(!)
(similarly for ideals in Δ^0_2 T-degrees)

Theorem

Let $\mathcal{C}$ be a Σ^0_3 ideal in the r.e. T-degrees. Then TFAE:

1. *there is a function F recursive in $\emptyset'$ which dominates all partial functions recursive in any member of the ideal $\mathcal{C}$,*
2. *there is a low T-upper bound for $\mathcal{C}$*

A slightly more general result.

Theorem

Let $\mathcal{C}$ be an ideal in Δ^0_2 T-degrees. Then TFAE:

1. *(a) $\mathcal{C}$ is contained in an ideal $\mathcal{A}$ which is generated by a sequence of sets $\{A_n\}_n$ such that the sequence is uniformly recursive in $\emptyset'$ and*
   (b) there is a function F recursive in $\emptyset'$ which dominates any partial function recursive in any set with T-degree in $\mathcal{A}$,
2. *there is a low T-upper bound for $\mathcal{C}$.*

Corollary

There is a low T-upper bound for the class $\mathcal{K}$ (the class of K-trivials).

Proof

Nies proved that the ideal (induced by) $\mathcal{K}$ is generated by its r.e. members and r.e. K-trivial sets induce a Σ^0_3 ideal in the r.e. T-degrees.
A.K. and Terwijn proved that there is a function F recursive in $\emptyset'$ which dominates all partial functions recursive in any member of $\mathcal{K}$ {Remark: Jump traceability of K-trivials is implicit in this result}.
Thus, Corollary follows from the previous Theorem.

Remark

Since every low set has a low PA set T-above it, low T-upper bounds which are PA are the most general case in this characterization (more about PA later).

The following lemma is the heart of the matter.

Lemma

Given a function F recursive in $\emptyset'$, there is a uniform way how to obtain from a $\emptyset'$-index of a set A with the property that any partial function recursive in A is dominated by F both a low set A^* and an index of lowness of A^* such that $A \leq_T A^*$,
i.e. there are recursive functions f, g such that if $\Phi_e(\emptyset')$ is total and equal to some set A so that any partial function recursive in A is dominated by F then $\Phi_{f(e)}(\emptyset')$ is a low set, $g(e)$ is its lowness index and $A \leq_T \Phi_{f(e)}(\emptyset')$.

Comment.

It is not possible, in general, to reach $A \leq_T A^*$ uniformly in an index of A, otherwise we would have a contradiction with a result of Nies (no effective listing of K-trivials together with their low indices).
Similarly, sets A^* cannot be, in general, obtained uniformly as r.e. sets.

Main idea

To combine forcing with Π^0_1 classes (like Low Basis Theorem) with coding sets into rich Π^0_1 classes, namely into subclasses of $\mathcal{P}A$.

A substantial use of $\{0,1\}$-valued DNR functions, i.e. *PA* sets.

Definition

Let $\mathcal{PA}(B)$ denote the class of $\{0,1\}$-valued B-DNR functions, i.e. the class of functions $f \in 2^\omega$ such that $f(x) \neq \Phi_x(B)(x)$ for all x . If B is $\emptyset$ we simply speak of $\mathcal{PA}$.

Definition (Simpson)

$\mathbf{b} << \mathbf{a}$ means that every infinite tree $T \subseteq 2^{<\omega}$ of degree $\leq \mathbf{b}$ has an infinite path of degree $\leq \mathbf{a}$.

Theorem (D. Scott and others)

The following conditions are equivalent:

1. $\mathbf{a}$ *is a degree of a* $\{0,1\}$*-DNR function*
2. $\mathbf{a} >> \mathbf{0}$
3. $\mathbf{a}$ *is a degree of a complete extension of PA*
4. $\mathbf{a}$ *is a degree of a set separating some effectively inseparable pair of r.e. sets.*

Remark

1. $\mathcal{PA}$ is a kind of a "universal" Π^0_1 class
2. $\{0,1\}$-valued DNR functions are also called PA sets and degrees $>> \mathbf{0}$ are called PA degrees.
3. (Simpson)
   (a) The partial ordering $<<$ is dense
   (b) $\mathbf{a} << \mathbf{b}$ implies $\mathbf{a} < \mathbf{b}$.

Definition

Let M be an infinite set and $\{m_0, m_1, m_2, \ldots\}$ be an increasing list of all members of M.

- If $f \in 2^\omega$ then by $Restr(f, M)$ we denote $g \in 2^\omega$ defined for all i by $g(i) = f(m_i)$
- Similarly, if $\mathcal{A} \subseteq 2^\omega$ then by $Restr(\mathcal{A}, M)$ we denote a class of functions $\{g : g = Restr(f, M) \ \wedge \ f \in \mathcal{A}\}$.

(*Idea: an analogue of a projection.*)

Lemma (A.K.)

- *For every Π^0_1 class $\mathcal{A} \subseteq \mathcal{PA}$ there is an infinite recursive set M such that if $\mathcal{A}$ is nonempty then $Restr(\mathcal{A}, M) = 2^\omega$, i.e. for every $g \in 2^\omega$ there is a function $f \in \mathcal{A}$ such that $Restr(f, M) = g$.*
- *For every $\Pi^{0,B}_1$ class $\mathcal{A} \subseteq \mathcal{PA}(B)$ there is an infinite recursive set M such that if $\mathcal{A}$ is nonempty then $Restr(\mathcal{A}, M) = 2^\omega$, where (an index of) M can be found uniformly from an index of $\mathcal{A}$, i.e. it does not depend on B.*

Remark

- This is basically Gödel incompleteness phenomenon
- It can be modified to a dynamic process, i.e. given an effective sequence of Σ^0_1 and Π^0_1 events, we can close (i.e. code) true Σ^0_1 ones while leaving open true Π^0_1 ones.

The Lemma is crucial for coding into members of (nonempty) Π^0_1 classes $\mathcal{A}$ which are subclasses of $\mathcal{PA}$.

We may

- code either an individual set C (by $Restr(\mathcal{A}, M) = \{C\}$)
- or nest another class $\mathcal{E} \subseteq 2^\omega$ (by $Restr(\mathcal{A}, M) = \mathcal{E}$)

Similarly with coding into members of nonempty $\Pi^{0,B}_1$ classes which are subclasses of $\mathcal{PA}(B)$.

Nesting in this way a $\Pi^{0,C}_1$ class into a $\Pi^{0,B}_1$ class we obtain $\Pi^{0,B\oplus C}_1$ class.

Idea of the proof of the main lemma (given a function F recursive in $\emptyset'$ and a set A with described properties).

An extremely simplified version : having a low index of A.

1. Code A into $\mathcal{PA}$, and get a $\Pi_1^{0,A}$ class (by $Restr(\mathcal{PA}, M) = A$, where M is an infinite recursive set used for coding)
2. Apply relativized Low Basis Theorem to get a member of the class.

A full version: we do not have a low index of A.

Missing low index of A is replaced by approximations provided by F to A'-questions. Since $(A^*)'$ has to be uniformly recursive in $\emptyset'$, our $\emptyset'$-construction of both A^* and $(A^*)'$ cannot change any decision about $(A^*)'(x)$ that it has already made. A wrong approximation to A'-question given by F leads eventually to a conflict with coding of A. We have to keep all our commitments about $(A^*)'(x)$ that we have already made and we have to start with a new coding strategy.

If A and F satisfy the given assumptions our method will guarantee that the approximations given by F will be correct from some point on, i.e. a coding strategy will eventually stabilize yielding $A \leq_T A^*$. Since we use Π^0_1 subclasses of $\mathcal{PA}$, we can always find a place for a new coding strategy (i.e. for coding an infinitary information). Here we substantially use the fact that (nonempty)

Π^0_1 subclasses of $\mathcal{PA}$ are rich

We use terms:
ω-extendability and F-extendability of a string in a tree (consider recursive trees yielding Π^0_1 classes $\subseteq \mathcal{PA}$ or A-recursive trees yielding $\Pi^{0,A}_1$ classes $\subseteq \mathcal{PA}(A)$).

Shortly, strings may be ω-good, $F(...)$-good etc.

We always have to keep

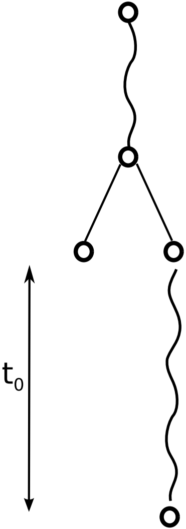
- ω-extendability of our strings in our recursive trees (trees for Π^0_1 subclasses of $\mathcal{PA}$)
- (only) F-extendability of these strings in A-recursive trees (trees for $\Pi^{0,A}_1$ subclasses of $\mathcal{PA}(A)$).

We explain the idea on a picture (first some notation).

Let G be a recursive function such that

$\lim_s G(\alpha, s) = F(\alpha)$.

We build an A-partial recursive function H, such that whenever we are in a real trouble, the value of H at such place will be greater than the value of F.
Since F has to eventually dominate H, from some point on there is no trouble at all and we win, i.e. a coding strategy will be stable and F-extendability will be, in fact, ω-extendability.



F-good, ω-bad (F doesn't know !)

still F-good, ω-bad (F doesn't know !)
$d_\emptyset$ = extendability, $H(..) = d_\emptyset > F(..)$

both F-bad, d_0, d_1 = extenadability
$F(..) > d_0, d_1$
F knows ! but A doesn't know !

Wait for t_0 with $G(.., t_0) > d_0, d_1$
Here A knows !
We can synchronize $\emptyset'$ and A-construction
We start a new coding strategy here
Note: a finite injury is behind, A doesn't know efectively where this happens

As a corollary of a result of Nerode and Shore there is an exact pair for the class $\mathcal{K}$ in Δ^0_2 T-degrees.

Question

1. Is there an exact pair for the class $\mathcal{K}$ in the r.e. T-degrees?
2. Is there a low exact pair for the class $\mathcal{K}$ in the Δ^0_2 T-degrees?

Comment

The desribed method (to produce a low T-upper bound) does not seem to be easily applicable to produce low exact pairs for ideals in question.
Example: there is no minimal pair of PA degrees below $\emptyset'$.

A very difficult and sharp question is the following.

Question

Is there a low 1-random set which is a T-upper bound for the class $\mathcal{K}$?

Observe, that for any T-upper bound given by a low 1-random set A
r.e. K-trivials $= \{B : B \text{ is r.e. } \& \ B \leq_T A\}$.

A very sharp property.
Even a much weaker question is open.

Question

Does for any K-trivial set A exist an incomplete 1-random set Y such that $A \leq_T Y$?
(Important case: r.e. K-trivials).

PART II

OTHER APPLICATIONS

Posner-Robinson

For every nonrecursive set Z : $\exists A (Z \oplus A \equiv_T A')$ (with $A \leq_T Z \oplus \emptyset'$).

It is obvious that in the above such A may be chosen to be a PA set (i.e. $A \in \mathcal{PA}$).

There is a direct and a slightly more general method for that.
Idea: Isolated paths through recursive trees are recursive.

Given a (nonempty) Π^0_1 class $\mathcal{A} \subseteq \mathcal{PA}$, and an infinite recursive set M with $Restr(\mathcal{A}, M) = 2^\omega$ and for each string effectively a Σ^0_1 question (or condition, like forcing the jump)

there has to exist a $\sigma \prec Z$ such that

- either for both $\sigma * i \quad (i = 0, 1)$ a Σ^0_1 event happens
- or for both $\sigma * i \quad (i = 0, 1)$ a Π^0_1 event happens

more precisely,
let $\mathcal{B}_i \subseteq \mathcal{A}$ $(i = 0, 1)$ be Π^0_1 classes such that
$Restr(\mathcal{B}_i, M) = \sigma * i * 2^\omega$, *(r.sp. $\sigma * i$ is coded in $\mathcal{A}$)*,

- either both $\mathcal{B}_i$ force a Σ^0_1 property
- or both $\mathcal{B}_i$ force a Π^0_1 property.

In case of a Σ^0_1 event we can take such σ longer than all Σ^0_1 witnesses in question. Then take $\mathcal{B}_i$ for $i \neq Z(|\sigma|)$, i.e. code a difference from Z.
R.sp. a piece of Z is coded into members of $\mathcal{A}$ and from that piece we can recognize what happened (whether Σ^0_1 or Π^0_1 case).

Definition

$A \leq_{LR} B$ if every set 1-random in B is also 1-random in A.

Definition

B is almost complete if $\emptyset'$ is K-trivial relative to B, i.e.
$\forall n \; (\; K^B(\emptyset' \upharpoonright n) \leq K^B(0^n) + O(1) \;)$

Lemma (Nies)

B is almost complete $\iff \emptyset' \oplus B \leq_{LR} B$

Thus, for Δ^0_2 sets: B is almost complete $\iff \emptyset' \leq_{LR} B$

Psedo-jump inversion.

Theorem (Jockusch,Shore,1983)

For every r.e. operator W, there is an r.e. set B such that $B \oplus W^B \equiv_T \emptyset'$.

Corollary (Nies)

There is an almost complete r.e. set $B <_T \emptyset'$.

Theorem

- *There is an almost complete PA set $A <_T \emptyset'$*
- *For every nonrecursive $Z \leq_T \emptyset'$, there is an almost complete PA set A such that $A \oplus Z \equiv_T \emptyset'$.*

The same technique as in a version of Posner-Robinson theorem for *PA* sets (above) applied to the r.e. operator obtained by relativizing a low for random r.e. set construction (or r.e. K-trivial set construction).
Thus, we have a cone avoidance by almost complete *PA* sets.

Coding into Π^0_1 classes of positive measure.

Theorem (A.K,1989)

- *There is an incomplete high 1-random set* $A <_T \emptyset'$.
- *For every set* B *r.e.a. in* $\emptyset'$ *and every nonrecursive* Δ^0_2 *set* C *there is a* Δ^0_2 *1-random set* A *such that* $A' \equiv_T B$ *and* $A \not\geq_T C$.

Theorem (Nies)

- *There is an almost complete 1-random set* $A <_T \emptyset'$
- *For every r.e. operator* W *there is a 1-random set* $A \leq_T \emptyset'$ *such that* $A \oplus W^A \equiv_T \emptyset'$.

Remark

In fact, the jump inversion technique in Theorem (A.K. 1989: high incomplete 1-random) yields immediately also a pseudo-jump inversion method and produces an incomplete 1-random almost complete set.

See: Simpson `http://www.math.psu.edu/~simpson`
paper: Mass Problems and Almost Everywhere Domination

where this is desribed.

(Other good paper of Simpson is about *LR*-reducibility, almost everywhere domination, a relation $\emptyset' \leq_{LR} A$, etc. is:
Almost Everywhere Domination and Superhighness.)

Definition

Let $\mathcal{K}_0$ denote the set of r.e. K-trivials which are T-below all almost complete 1-random sets.

Theorem (Hirschfeldt, Miller)

For every Σ^0_3 null class $\mathcal{C}$ there is a nonrecursive r.e. set which is T-below all 1-random sets in $\mathcal{C}$.

Corollary (Hirschfeldt, Miller)

$\mathcal{K}_0$ (a subclass of the r.e. members of $\mathcal{K}$) contains also nonrecursive r.e. sets.

Sets in $\mathcal{K}_0$ are ML-noncuppable, i.e. for such sets A, $A \oplus Z <_T \emptyset'$ for all Δ^0_2 1-random sets $Z <_T \emptyset'$. (Nies was the first proving the existence of a ML-noncuppable r.e. K-trivial set).

Known so far

- Every DNR function $\leq_T \emptyset'$ bounds a nonrecursive r.e. set
- Even for every $\emptyset''$-sequence of $\emptyset'$-indices of DNR functions there is a nonrecursive r.e. set which is T-below all of them
- (Hirschfeldt, Miller) For every Σ^0_3 null class $\mathcal{C}$ there is a nonrecursive r.e. set which is T-below all 1-random sets in $\mathcal{C}$
- there is no nonrecursive r.e. set T-below all almost complete *PA* sets
  (and there is no nonrecursive r.e. set T-below all Δ^0_2 almost complete *PA* sets)

Thus, there is no cone avoidance by almost complete 1-random sets, since at least some (all ?) nonrecursive K-trivials have all almost complete 1-random sets T-above. Recall: there is a cone avoidance by almost complete PA sets (even by Δ^0_2 ones).

Question

Is there a cone avoidance by almost complete r.e. sets ?

Other open questions

- Is $\mathcal{K}_0$ equal to r.e. members of $\mathcal{K}$?
- Are there minimal pairs of r.e. almost complete sets ?
- Can a K-trivial set be ML-cuppable?

Comment

Many obstacles in solving the above questions concerning 1-randomness are connected with a problem of coding an information into 1-random sets.

While we can code an infinitary information into *PA* sets (or into members of Π^0_1 subclasses of $\mathcal{PA}$),

coding an information into 1-random sets (or into members of Π^0_1 classes of positive measure) is less powerful and it is still not completely understood.

The paper about low upper bounds of ideals is submitted to a journal, a preprint can be found at

`http://math.berkeley.edu/~slaman/papers`

Thank you