# $\Pi^0_1$ CLASSES WITH COMPLEX ELEMENTS.

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**Abstract.** An infinite binary sequence is *complex* if the Kolmogorov complexity of its initial segments is bounded below by a computable function. We prove that a $\Pi^0_1$ class $P$ contains a complex element if and only if it contains a wtt-cover for the Cantor set. That is, if and only if for every real $Y$ there is an $X$ in the $P$ such that $X \geqslant_{wtt} Y$. We show that this is also equivalent to the $\Pi^0_1$ class's being large in some sense. We give an example of how this result can be used in the study of scattered linear orders.

**§1. Introduction.** There has been interest in the literature over many years in studying various notions of the size of subclasses of $2^\omega$. In this paper we have tried to generalise and consolidate some of these ideas. We investigate a notion of size that has appeared independently in [1] and [5], namely the notion of a *computable perfect* class (*computably growing* in [5] and *non-uphi* in [1]). It is a relatively straightforward and natural computability theoretic notion and we give the formal definition in Section 2, but the idea is that a closed subclass of $2^\omega$ is computably perfect if there is a computable witness to the fact that it has no isolated points. Equivalently, there is a computable witness to the fact that it has continuum many elements. We declare to be *diminutive* those classes that fail to contain a computably perfect subclass.

We are particularly concerned with effectively closed sets of reals, or $\Pi^0_1$ classes, and we give a very neat characterisation of diminutive $\Pi^0_1$ classes - *viz.* a $\Pi^0_1$ class is diminutive if and only if it does not contain a *complex* element. A real is complex if there is a computable lower bound for the Kolmogorov complexity of its initial segments. That is, a real is complex if there is a computable function $f$ such that for all $n$

$$C(X\restriction n) \geqslant f(n),$$

where $C(X\restriction n)$ is the plain Kolmogorov complexity of $X\restriction n$.

In Section 3 we show that some commonly studied subclasses of $2^\omega$ are diminutive. Countable classes, *thin* classes and *small* classes, among others are all diminutive. Classes of positive measure or Hausdorff dimension, among other examples of interest, are not diminutive.

To the extent that this theorem connects the existence of a particular kind of real (a complex one) in a $\Pi^0_1$ class with a global property of the $\Pi^0_1$ class

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(computable perfection), the result is reminiscent of Kučera's theorem that a $\Pi^0_1$ class of reals contains a Martin-Löf random if and only if the class has positive measure.

These classes are also useful in that they furnish us with the strongest possible generalisation of the Kučera-Gács Theorem. This theorem proved independently by Kučera [7] and Gács [4] states that every real number is wtt-computable from some random real. In fact, it is an easy consequence of other results of Kučera's that every $\Pi^0_1$ class of positive measure contains a wtt-cover for the class of reals (i.e. every real is wtt-computable from some element of any non-null $\Pi^0_1$ class). The Kučera-Gács theorem has been generalised in by Hertling in [5] where it is shown that every computably perfect $\Pi^0_1$ class contains such a wtt-cover for the reals. In [9] Reimann proves that every $\Pi^0_1$ class of positive Hausdorff dimension contains a computably perfect $\Pi^0_1$ subclass, and hence a wtt-cover.

Our result here establishes the converse of result of Hertling and provides the Reimann result as an easy corollary. We prove in Section 2.1 that a $\Pi^0_1$ class contains a complex element if and only if it contains a wtt-cover for the reals. In other words, the $\Pi^0_1$ classes for which the Kučera-Gács theorem holds are exactly the ones containing complex elements (which are exactly the ones that are not diminutive in the above sense). As all $\Pi^0_1$ classes of positive Hausdorff dimension have complex elements, Reimann's result follows.

In Section 4 we give an example of diminutive classes in the study of scattered linear orders. A linear order is *scattered* if it does not contain a suborder isomorphic to $\mathbb{Q}$. It is well known that if a countable linear order is computable, then the class of (characteristic functions of) initial segments of the order forms a $\Pi^0_1$ class of reals. In [8] it is shown that if a computable linear order has a complex initial segment, then the linear order is not scattered. Our theorem extends and sharpens this result, and we give a precise characterisation of computable linear orders that do not have complex initial segments. These we call *weakly scattered linear orders*. They are related to scattered linear orders in precisely the way that diminutive $\Pi^0_1$ classes are related to countable $\Pi^0_1$ classes. It is an immediate application of the main theorem of Section 2.13 that the wtt-degree spectrum of initial segments of a computable linear order is maximal if and only if the linear order is not weakly scattered.

This method of applying the main result of Section 2.13 should be easily generalisable and give results about the wtt-degree spectra of other computable structures and relations.

**1.1. Notation and basic definitions.** We first make explicit a few, mostly standard, definitions and notations. Undefined notation follows [11].

Basic notation.

- *We use $\omega$ to denote the set of natural numbers; $2^{<\omega}$ to denote the set of finite binary strings; and $2^\omega$ is the set of infinite binary sequences. $\omega^\omega$ is the collection of all functions from $\omega$ to $\omega$. $\sigma$ and $\tau$ will usually denote elements of $2^{<\omega}$ and $X$ and $Y$ elements of $2^\omega$. $\lambda$ denotes the empty string and $\sigma i$ ($i \in \{0,1\}$) denotes the concatenation of $\sigma$ with $\langle i \rangle$. $|\sigma|$ is the length of $\sigma$. $\sigma \supseteq \tau$ implies that $\sigma$ extends $\tau$ and similarly for $X \supset \sigma$ with $X \in 2^\omega$.*

*If* $\sigma \in 2^{<\omega}$*, then* $[\sigma] = \{X \in 2^\omega : X \supset \sigma\}$ *and if* $S \subseteq 2^{<\omega}$*, then* $[S]$ *denotes the class* $\bigcup\{[\sigma] : \sigma \in S\}$*.*

- *The standard topology will be assumed on* $2^\omega$*, the basic open sets of which are of the form* $[\sigma]$ *for* $\sigma \in 2^{<\omega}$*.*
- *If* $C \subseteq 2^\omega$*, then the set of* extendible nodes of $C$ *is the set*

$$\mathrm{Ext}(C) = \{\sigma \in 2^{<\omega} : \exists X \in C X \supset \sigma\}.$$

*The* set of branching nodes of $C$ *is the set*

$$\mathrm{Br}(C) = \{\sigma \in \mathrm{Ext}(C) : \sigma 0 \in \mathrm{Ext}(C) \wedge \sigma 1 \in \mathrm{Ext}(C)\}.$$

*The* set of branching levels of $C$ *is the set*

$$\mathrm{Brl}(C) = \{|\sigma| : \sigma \in \mathrm{Br}(C)\}$$

*If* $X \in C$*, then the* branching level set along $X$ *is the set*

$$\mathrm{Br}_X(C) = \{n \in \omega : X\restriction n \in \mathrm{Br}(C)\}.$$

*If* $\sigma \in \mathrm{Ext}(C)$ *then* $\mathrm{Br}_\sigma(C)$ *is defined similarly.*

- *If* $C$*,* $D \subseteq \omega^\omega$*, and if*

$$\forall X \in C \exists Y \in D X \geqslant_T Y,$$

*then we say that* $D$ *is* Muchnik reducible to $C$ *and write* $C \geqslant_w D$*. Surveys of Muchnik reducibility can be found in [2] and [10].*

Definition 1.1. If $X \subseteq \omega$, the *principal function of* $X$ $p_X$ is the function

$$p_X(n) = (n+1)^{\text{th}} \text{ least element of } X.$$

Definition 1.2. $X \subseteq \omega$ is said to be *hyperimmune* if $p_X$ is not dominated by any computable function.

Definition 1.3. $P \subseteq 2^\omega$ is a $\Pi^0_1$ *class* if it is the collection of paths through some computable tree of binary strings. Equivalently, $P = \{X \in 2^\omega : \forall n R(n, X)\}$ where $R$ is a computable predicate on $\omega \times 2^\omega$. Equivalently, $P$ is $\Pi^0_1$ if it is of the form $2^\omega \setminus [S]$ where $S$ is a c.e. subset of $2^{<\omega}$. If $S_s$ is the enumeration of $S$ at stage $s$, then $P_s := 2^\omega \setminus [S_s]$, and $P = \bigcap_s P_s$.

$\Pi^0_1$ classes can be defined other spaces including $\omega^\omega$ and $\mathbb{R}$, however for the purposes of this paper, they will always be non-empty subsets of $2^\omega$.

## §2. Computably perfect classes.

Definition 2.1. $C \subseteq 2^\omega$ is *computably perfect* if $C$ is closed and there is a computable and strictly increasing function $f$ such that for every $n \in \omega$ and every $\sigma \in \mathrm{Ext}(C)$ of length $f(n)$, there exist at least two extensions of $\sigma$ in $\mathrm{Ext}(C)$ of length $f(n+1)$.

Observation 2.2. It is straightforward to show that $C$ is computably perfect if and only if it is closed and there is a computable function $g$ such that for all $n$ and all $\sigma \in \mathrm{Ext}(C)$ of length $n$, there exists at least two extensions of $\sigma$ of length $g(n)$.

Computably perfect classes have been called computably growing classes in [5] and non-uphi classes in [1].

OBSERVATION 2.3. It is straightforward to show that $C$ is computably perfect if and only if there is a computable function $f$ such that for all $X \in C$ and for all $n \in \omega$

$$||\mathrm{Br}_{X\restriction f(n)}(C)|| \geqslant n.$$

LEMMA 2.4. *If a $\Pi^0_1$ class $P$ contains a computably perfect subclass, then $P$ also contains a computably perfect $\Pi^0_1$ subclass.*

PROOF. Let $C$ be any computably perfect subclass of $P$ witnessed by the computable function $f$. Define the set $S$ to be

$$\{\sigma : \exists s \exists n \leqslant s \sigma \in \mathrm{Ext}(P_s) \wedge |\sigma| = f(n) \wedge \exists! \tau \in \mathrm{Ext}(P_s)[\tau \supseteq \sigma \wedge |\tau| = f(n+1)]\}.$$

Then $P \setminus [S]$ is $\Pi^0_1$, contains $C$ and is computably perfect. ⊣

We will make use of the following well-known idea from algorithmic complexity theory.

DEFINITION 2.5. $X \in 2^\omega$ is *complex* if there is a computable function $f$ such that for all $n$

$$C(X\restriction f(n)) \geqslant n.$$

Here $C(X\restriction m)$ denotes the plain Kolmogorov complexity of the string $X\restriction m$. Alternatively, we could have used the prefix-free complexity $K(X\restriction m)$. We may also equivalently define $X$ to be complex if there exists a computable $f$ such that for all $n$ $C(X\restriction n) \geqslant f(n)$.

The following two definitions are standard.

DEFINITION 2.6. If $X, Y \in 2^\omega$, then $Y$ is weak truth-table reducible to $X$ ($X \geqslant_{wtt} Y$) if $X \geqslant_T Y$ and there is a total computable function $f$ such that for all $n$, the use of $X$ in calculating $Y(n)$ is bounded by $f(n)$.

DEFINITION 2.7. If $f \in \omega^\omega$, then $f$ is *DNR* if $\forall n f(n) \neq \{n\}(n)$, where $\{n\}$ is the $n^{\text{th}}$ partial computable function.

The major result we will be using with regards to complexity is Theorem 6 in [12]:

LEMMA 2.8. *For $A \in 2^\omega$, $A$ is complex if and only if $A$ wtt-computes a DNR function.*

There are of course many $\Pi^0_1$ classes that contain complex elements - $2^\omega$ obviously. The following two lemmas give two less trivial examples.

DEFINITION 2.9. The *effective Hausdorff dimension of* $X \in 2^\omega$ $\dim^1_{\mathcal{H}}(X)$ can be defined to be $\liminf_n C(X\restriction n)/n$. For a class $C \subseteq 2^\omega$,

$$\dim^1_{\mathcal{H}}(C) = \sup\{\dim^1_{\mathcal{H}}(X) : X \in C\}.$$

For a $\Pi^0_1$ class $P$ the effective Hausdorff dimension of $P$ is equal to its classical Hausdorff dimension $\dim_{\mathcal{H}}(P)$. A good survey of Hausdorff dimension, both effective and classical, as it applies to the space $2^\omega$ is given in [9].

LEMMA 2.10. *If $P$ is a $\Pi^0_1$ class and if $dim_{\mathcal{H}}(P) > 0$, then $P$ has a complex element.*

PROOF. If $X \in 2^\omega$ is not complex, then for all rational $s > 0$ $\exists^\infty n C(X \restriction n) < sn$. Therefore, for all such $s$, $\liminf_n C(X \restriction n)/n < s$ and thus $\dim^1_{\mathcal{H}}(X) = 0$. So, if every element of $P$ is noncomplex,

$$\dim_{\mathcal{H}}(P) = \dim^1_{\mathcal{H}}(P) = \sup\{\dim^1_{\mathcal{H}}(X) : X \in P\} = 0$$

⊣

LEMMA 2.11. *If $P$ is $\Pi^0_1$, and if $P \geqslant_w DNR$, then $P$ has a complex element.*

PROOF. Every $\Pi^0_1$ class contains a hyperimmune-free element (see [6]), and for any such element $X$ and any $Y \leqslant_T X$, we also have that $Y \leqslant_{wtt} X$. Thus if $X$ computes an element of DNR, it must be complex by Lemma [12].

⊣

### 2.1. The Main theorem.

DEFINITION 2.12. $\mathcal{C} \subseteq 2^\omega$ is a *wtt-cover for* $2^\omega$ if every element of $2^\omega$ is wtt-reducible to an element of $\mathcal{C}$.

THEOREM 2.13. *For any $\Pi^0_1$ class $P$ the following are equivalent:*

1. *$P$ contains a wtt-cover for $2^\omega$,*
2. *$P$ contains a complex element $X$,*
3. *$P$ contains a computably perfect $\Pi^0_1$ subclass containing $X$.*

PROOF.
*1 ⇒ 2.*
$P$ must contain an element that wtt-computes an element of a DNR set, and thus by Lemma [12], must contain a complex element.

*2 ⇒ 3.*
Let $f$ be computable and such that $C(X \restriction f(n)) \geqslant n$ for all $n$. Define, for parameter $u \in \omega$ a function $g$ as follows.

1. $g(0) = f(0)$
2. $g(n+1) = f(2g(n) + u)$.

We will use $u$ for bookkeeping purposes at the end of the argument.

We have immediately that for all $n$,

$$C(X \restriction g(n+1)) = C(X \restriction f(2g(n) + u)) \geqslant 2g(n) + u$$

and from basic principles that

$$C(X \restriction g(n)) < 2g(n) + d$$

for some fixed constant $d$.

To get a contradiction, suppose now that $X$ is not contained in any computably perfect $\Pi^0_1$ subclass of $P$. Consider the $\Pi^0_1$ class $Q \subseteq P$ constructed in the following way. We let $Q_0 = P_0 = 2^\omega$, $S_0 = \emptyset$, and for each $s$ let

$$S_{s+1} = \{\sigma \in \mathrm{Ext}(Q_s) : \exists n < s |\sigma| = g(n) \wedge \exists! \tau \in \mathrm{Ext}(Q_s)[\tau \supseteq \sigma \wedge |\tau| = g(n+1)]\},$$

and $Q_{s+1} = (P_{s+1} \cap Q_s) \setminus [S_{s+1}]$.

$Q = \bigcap_s Q_s$ is either computably perfect (witnessed by $g$) or empty. In either case $X \notin Q$ and so $X$ must extend some element of $S_{s+1}$ for some least $s$. Let $n < s$ be such that $X\upharpoonright g(n) \in S_{s+1}$. There is thus a unique extension of $X\upharpoonright g(n)$ in $\mathrm{Ext}(Q_s)$ of length $g(n+1)$, namely $X\upharpoonright g(n+1)$.

Now consider a machine $M$ that acts as follows. Taking input $\tau \in 2^{<\omega}$ of length $g(m)$ for some $m$, $M$ waits for a stage $s$ such that $\tau \in S_{s+1}$. If such a stage exists, then $M$ outputs the unique string on $\mathrm{Ext}(Q_s)$ extending $\tau$ of length $g(m+1)$. Thus, for $n$ and $X$ as above, $M(X\upharpoonright g(n)) = X\upharpoonright g(n+1)$ and $C(X\upharpoonright g(n+1)) \leqslant C(X\upharpoonright g(n)) + \mathcal{O}(1)$.

The constant term $\mathcal{O}(1)$ depends uniformly on a $\Pi^0_1$ index for $Q$, so there is a computable function $k$ such that, for any $\Pi^0_1$ index $e$ for $Q$,

$$C(X\upharpoonright g(n+1)) \leqslant C(X\upharpoonright g(n)) + k(e).$$

Furthermore, there is a computable function $\varphi$, that will give such an index for $Q$ uniformly in the parameter $u$. Using the recursion theorem we choose a $u$ such that the $\Pi^0_1$ class with index $u$ is equal to the $\Pi^0_1$ class whose index is $\varphi(d + k(u))$. Then the $Q$ that is built using parameter $d + k(u)$ is equal to the $\Pi^0_1$ class with index $u$. Thus, if $g$ is defined using the parameter $d + k(u)$,

$$C(X\upharpoonright g(n+1)) \leqslant C(X\upharpoonright g(n)) + k(u).$$

Therefore, for such a $u$ and $n$,

$$2g(n) + d + k(u) \leqslant C(X\upharpoonright g(n+1)) \leqslant C(X\upharpoonright g(n)) + k(u) < 2g(n) + d + k(u),$$

giving the required contradiction.

*3 ⇒ 1.*

Let $P$ be computably perfect, witnessed by a computable function $f$. If $X \in 2^\omega$ define a sequence of nodes $\sigma_i$, as follows: let $\sigma_0 = \lambda$ and let $\sigma_{i+1}$ be the leftmost extendible node of $P$ of length $f(i+1)$ extending $\sigma_i$ if $X(i) = 0$, and the rightmost such node of length $f(i+1)$ if $X(i) = 1$. Such nodes are always distinct as $\sigma_i$ has at least two extensions of length $f(i+1)$ by assumption. Let $Y \in P$ be $\bigcup_i \sigma_i$.

To recover $X(n)$ from $Y$, wait for a stage $s$ such that $Y\upharpoonright f(n+1)$ is the leftmost or rightmost node extending $Y\upharpoonright f(n)$ in $P_s$. Such a stage will always occur as $Q$ is $\Pi^0_1$. The use of $Y$ in the computation of $X(n)$ is bounded by $f(n+1)$ and hence $Y \geqslant_{wtt} X$.

⊣

Corollary 2.14. *The following classes all contain wtt-covers for* $2^\omega$*:*

1. *(Kučera-Gács) the class of Martin-Löf randoms,*
2. *any* $\Pi^0_1$ *class of positive measure,*
3. *([9], Thm 3.25) any* $\Pi^0_1$ *class with positive Hausdorff dimension,*
4. *any* $\Pi^0_1$ *class* $P \geqslant_w DNR$*.*

Proof. The randoms form a $\S^0_2$ class - a union of $\Pi^0_1$ classes . Any one of these component $\Pi^0_1$ classes contains a wtt-cover for $2^\omega$ as every random is complex. Any $\Pi^0_1$ class of positive measure contains a random, and hence a complex. Lemmas 2.10 and 2.11 complete the proof. ⊣

## §3. Diminutive classes.

DEFINITION 3.1. $C \subseteq 2^\omega$ is *diminutive* if it is closed and does not contain a computably perfect subclass.

There are many examples of diminutive classes already existing in the literature. The simplest examples are countable $\Pi^0_1$ classes which have been studied in [13] and elsewhere. We give some more examples now.

DEFINITION 3.2. A $\Pi^0_1$ class $P$ is *small* if $\mathrm{Br}(P)$ is hyperimmune. It is observed in [2] that it is equivalent that the *branching level set*

$$\mathrm{Brl}(P) = \{n \in \omega : \exists \sigma \in \mathrm{Br}(P) |\sigma| = n\}$$

is hyperimmune.

DEFINITION 3.3. A $\Pi^0_1$ class $P$ is *e.p.h.i* (everywhere pathwise hyperimmune) if for all $X \in P$ the set $\mathrm{Br}_X(P) = \{n : X{\upharpoonright}n \in \mathrm{Br}(P)\}$ is hyperimmune.

DEFINITION 3.4. A $\Pi^0_1$ class $P$ is a *Jockusch-Soare class* (js-class) if for all distinct $X, Y \in P X \not\geqslant_T Y$.

DEFINITION 3.5. A $\Pi^0_1$ class $P$ is *thin* if every $\Pi^0_1$ subclass of $P$ is the intersection of a clopen subclass of $2^\omega$ with $P$. That is, every $\Pi^0_1$ subclass of $P$ is clopen in the relative topology.

Small classes are special instances of e.p.h.i. classes, as if the branching level set is hyperimmune, then certainly the set along any path will also be hyperimmune. We prove next that every thin class is e.p.h.i. and that every e.p.h.i. class is diminutive. We then show that every js-class is diminutive.

LEMMA 3.6. *No thin $\Pi^0_1$ class is computably perfect.*

PROOF. The proof is very similar to Simpson's proof that all thin $\Pi^0_1$ classes have zero measure. We prove the contrapositive. Suppose a $\Pi^0_1$ class $P$ is computably perfect, witnessed by the computable function $f$. Without losing generality we can assume that for all $s$ $P_s$ is computably perfect witnessed by $f$.

Define a computable double sequence of elements of $2^{<\omega}$ as follows:

$\sigma_{1,s}$ = the rightmost string in $\mathrm{Ext}(P_s)$ of length $f(1)$
$\sigma_{i+1,s}$ = the rightmost string in $\mathrm{Ext}(P_s)$ of length $f(i+1)$
strictly to the left of $\sigma_n$

To prove that $\sigma_{i,s}$ exists for all $i$ and $s$ we use a simple induction to prove that for all $i > 0$ and $s \geqslant 0$ there is a $\tau \in \mathrm{Ext}(P_s)$ of length $f(i)$ such that $\tau$ is strictly to the left of $\sigma_{i,s}$. If $\tau$ is the rightmost such string, then $\sigma_{i+1}$ will be the rightmost element in $\mathrm{Ext}(P)$ of length $f(i+1)$ extending $\tau$.

*Base case:* There are at least two elements of $\mathrm{Ext}(P)$ of length $f(1)$, so there must be a $\tau \in \mathrm{Ext}(P)$ of length $f(1)$ strictly to the left of $\sigma_1$.

*Induction:* Suppose that $\tau$ is the rightmost element in $\mathrm{Ext}(P_s)$ of length $f(i)$ strictly to the left of $\sigma_{i,s}$. There must be at least two extensions of $\tau$ in $\mathrm{Ext}(P_s)$ of length $f(i+1)$. Therefore there must be a $\tau' \in \mathrm{Ext}(P_s)$ of length $f(i+1)$ strictly to the left of $\sigma_{i+1}$.

If we let $\sigma_i = \lim_s \sigma_{i,s}$ for each $i$, then the strings $\sigma_i$ are pairwise incomparable and form a leftwards monotone sequence. Therefore the class

$$S = \{X \in 2^\omega : \exists i \exists s X \restriction f(i) \text{ is to the right of } \sigma_{i,s}\}$$

is not closed (it does not contain its leftmost limit point). However $S$ is $\Sigma^0_1$ by inspection, and so $P \setminus S$ is a non-clopen $\Pi^0_1$ subclass of $P$ and $P$ is not thin. ⊣

Corollary 3.7. *No thin $\Pi^0_1$ class contains a computably perfect subclass.*

Proof. By Lemma 2.4, if a thin $\Pi^0_1$ class $T$ contained a computably perfect subclass, it would contain a computably perfect $\Pi^0_1$ subclass $T'$. Any $\Pi^0_1$ subclass of a thin class is also thin so $T'$ would be a thin and computably perfect, contradicting the theorem. ⊣

This is enough to show that all thin classes are diminutive, but we can go further:

Theorem 3.8. *Every thin $\Pi^0_1$ class is e.p.h.i.*

Proof. Suppose a $\Pi^0_1$ class $P$ is thin and not e.p.h.i. and let $f$ be a computable function such that for some element $X$ of $P$ $f$ dominates $\mathrm{Br}_X(P)$. Consider the set of all such elements:

$$P' = \{Y \in P : \forall n ||\mathrm{Br}_{Y\restriction n}(P)|| \geqslant f(n)\}.$$

$P'$ is a then non-empty $\Pi^0_1$ subclass of $P$. As $P$ is thin, $P'$ is a clopen subclass of $P$ and there is a fixed constant $c$ such that for all $X \in P'$ the sets $\mathrm{Br}_X(P)$ and $\mathrm{Br}_X(P')$ differ by at most $c$ elements. Thus for some finite adjustment $f'$ of $f$, and all $X \in P'$

$$\forall n ||\mathrm{Br}_{X\restriction n}(P')|| \geqslant f'(n),$$

and thus $f'$ witnesses the fact that $P'$ is computably perfect. Thus by Lemma 3.6 $P'$ is not thin and hence neither is $P$. ⊣

Theorem 3.9. *E.p.h.i. classes contain no computably perfect subclass.*

Proof. Suppose $P$ is e.p.h.i. and $P' \subseteq P$ is computably perfect witnessed by $f$. Then for all $X \in P$ and all $n \in \omega$, there is a branching node on $P'$ between $X \restriction f(n)$ and $X \restriction f(n+1)$. As the branching nodes of $P'$ form a subset of the branching nodes of $P$, $f$ also witnesses the fact that $P$ is not e.p.h.i.. ⊣

Finally we show that no js-class contains a complex element.

Theorem 3.10. *Js-classes contain no complex element.*

Proof. Let $P$ be a js-class and suppose it contains a complex element. Let $\sigma$ be the shortest branching node on $P$ and consider $P^0 = \{X \in P : X \supset \sigma 0\}$ and $P^1 = \{X \in P : X \supset \sigma 1\}$. Without losing generality, suppose $P^0$ contains a complex element. Then $P^0$ contains a wtt-cover for $2^\omega$, and hence contains elements that compute elements of $P^1$, contradicting the fact that $P$ is a js-class. ⊣

**§4. Weakly scattered linear orderings.** If $\mathcal{L} = \langle \omega, <_{\mathcal{L}} \rangle$ is a countable linear order, we will denote the set of (characteristic functions of) initial segments of $\mathcal{L}$ by

$$\mathrm{Seg}(\mathcal{L}) = \{f : \forall n, m[(f(n) = 1 \wedge m <_{\mathcal{L}} n) \Rightarrow f(m) = 1]\}.$$

DEFINITION 4.1. A countable linear order is *scattered* if it does not contain a suborder isomorphic to $\mathbb{Q}$.

The following is well-known.

THEOREM 4.2. *A linear order $\mathcal{L}$ is scattered if and only if it Seg($\mathcal{L}$) does not contain a perfect subclass. In fact if Seg($\mathcal{L}$) contains a perfect subclass $C$, then the branching level set $S := \mathrm{Brl}(C) = \{|\sigma| : \sigma \in \mathrm{Br}(C)\}$ is isomorphic to $\mathbb{Q}$.*

PROOF. First assume that $\mathrm{Seg}(\mathcal{L})$ contains a perfect subclass $C$. We show that $S$ is a dense linear order with no endpoints.

Let $\sigma$ be an arbitrary element of $\mathrm{Br}(\overline{C})$. Let $\sigma^0, \sigma^1 \in \mathrm{Br}(\overline{C})$ such that $\sigma^0 \supsetneq \sigma 0$ and $\sigma^1 \supsetneq \sigma 1$. Both exist as $\overline{C}$ is perfect. Thus there is an $X \in \mathrm{Seg}(\mathcal{L})$ such that $X(|\sigma|) = 0$ and $X(|\sigma^0|) = 1$, so $|\sigma^0| <_{\mathcal{L}} |\sigma|$. Similarly, there is an initial segment $Y$ such that $Y(|\sigma|) = 1$ and $Y(|\sigma^1|) = 0$ and so $|\sigma| <_{\mathcal{L}} |\sigma^1|$. So $S$ has no endpoints.

To show that $S$ is dense, let $\sigma$ and $\tau$ be arbitrary elements of $\mathrm{Br}(\overline{C})$ with $|\sigma| <_{\mathcal{L}} |\tau|$. We divide into two cases:

*Case 1.* $\sigma$ and $\tau$ are incomparable. Let $\gamma$ be any branching node extended by both $\sigma$ and $\tau$. By an argument similar to above we must have $\sigma <_{\mathcal{L}} \gamma <_{\mathcal{L}} \tau$.

*Case 2.* As $|\sigma| <_{\mathcal{L}} |\tau|$, we must have either $\sigma \supseteq \tau 0$ or $\tau \supseteq \sigma 1$. Assume first that $\sigma \supseteq \tau 0$. Let $\gamma \in \mathrm{Br}(C)$ such that $\gamma \supseteq \sigma 1$. Then $\gamma(|\sigma|) = 1$ and $\gamma(|\tau|) = 0$ and there is an $X \in \mathrm{Seg}(C)$ such that $X(|\sigma|) = 1$, $X(|\gamma|) = 1$ and $X(|\tau|) = 0$. Therefore, $|\sigma| <_{\mathcal{L}} |\gamma| <_{\mathcal{L}} |\tau|$. We argue similarly if $\tau \supseteq \sigma 1$.

Conversely, suppose $\mathcal{L}$ contains a suborder $\mathcal{M}$ isomorphic to $\mathbb{Q}$. Then $\mathcal{M}$ has uncountably many initial segments, and hence so does $\mathcal{L}$. As $\mathrm{Seg}(\mathcal{L})$ is closed, it contains a perfect subclass.

⊣

We now weaken the notion of scatteredness in a way that allows us to replace *perfection* in the previous theorem with *computable perfection*.

DEFINITION 4.3. A countable linear order $\mathcal{L} = \langle \omega, <_{\mathcal{L}} \rangle$ is *weakly scattered* if there does not exist an $S \subseteq \omega$ and a computable function $f$ such that for all $n \in \omega$,

$$\forall a, b \in S \cap [0, n] \exists x, y, z \in S \cap [0, f(n)] x <_{\mathcal{L}} a <_{\mathcal{L}} y <_{\mathcal{L}} b <_{\mathcal{L}} z.$$

Notice that such an $S$ is isomorphic to $\mathbb{Q}$, and so any scattered linear order is necessarily weakly scattered. Furthermore, if we are given $a, b \in S$, the problem of finding $x, y, z$ such that $x <_{\mathcal{L}} a <_{\mathcal{L}} y <_{\mathcal{L}} b <_{\mathcal{L}} z$ is the problem of finding certain branching nodes on $\mathrm{Seg}(\mathcal{L})$. This suggests a connection with diminutive $\Pi^0_1$ classes and we make this explicit now.

THEOREM 4.4. *A countable linear order $\mathcal{L} = \langle \omega, <_{\mathcal{L}} \rangle$ is not weakly scattered if and only if Seg($\mathcal{L}$) contains a computably perfect subclass.*

Proof. Suppose $\mathcal{L}$ is not weakly scattered, witnessed by the suborder $\mathcal{S} = \langle S, \leqslant_{\mathcal{L}} \rangle$ isomorphic to $\mathbb{Q}$, and computable function $f$. Let $P = \mathrm{Seg}(\mathcal{S})$, $n \in \omega$, and $\sigma \in \mathrm{Ext}(P)$ of length $n$. Let $a = \max_{\mathcal{L}}\{c < n : \sigma(c) = 1\}$ ($-\infty$ if empty) and $b = \min_{\mathcal{L}}\{c < n : \sigma(c) = 0\}$ ($\infty$ if empty). Thus $a <_{\mathcal{L}} b$. As $f$ witnesses the fact that $\mathcal{L}$ is not weakly scattered, there is an $y \in S \cap [0, f(n)]$ such that $a <_{\mathcal{L}} y <_{\mathcal{L}} b$. So there are two initial segments $X$, $Y$ of $\mathcal{S}$, both extending $\sigma$, such that $X(y) = 0$ and $Y(y) = 1$. Thus $\sigma$ has at least two extensions of length $f(n)$ in $\mathrm{Ext}(P)$ and by Observation 2.2, $P$ is computably perfect.

Conversely, suppose $\mathrm{Seg}(\mathcal{L})$ contains a computably perfect subclass $P$. As before, $S = \{|\sigma| : \sigma \in \mathrm{Br}(P)\}$ will be the domain of the suborder isomorphic to $\mathbb{Q}$. Let $n \in \omega$ and $\sigma, \tau \in \mathrm{Br}(P)$ be arbitrary and without losing generality suppose $|\sigma| <_{\mathcal{L}} |\tau|$. Arguing as in Theorem 4.2, $f$ allows us to find upper bounds for the minimum lengths of branching nodes extending $\sigma$ and $\tau$ and thus allows us to find an upper bound for some $x, y, z$ such that $x <_{\mathcal{L}} |\sigma| <_{\mathcal{L}} y <_{\mathcal{L}} |\tau| <_{\mathcal{L}} z$. So from $f$ we can easily compute a witness to the fact that $\mathrm{Seg}(\mathcal{L})$ is not weakly scattered.

⊣

Theorem 4.5. *There is a weakly scattered linear ordering that is not scattered.*

Proof. We create such an order $\mathcal{L}$ using a priority argument. Let $\eta$ be the order type of $\mathbb{Q}$. $\mathcal{L}$ will be an $\eta$-order of finite linear orders or *blocks* $B_i$. Any element of a block may act as an index for the block. We write $B_i <_{\mathcal{L}} B_j$ if every element of $B_i$ is $<_{\mathcal{L}}$ every element of $B_j$ (equivalently if some element of $B_i$ is $<_{\mathcal{L}}$ some element of $B_j$) .

We enumerate elements of $\omega$ in numerical order into $\mathcal{L}$ in stages. As usual $\mathcal{L}_s = \langle L_s, <_{\mathcal{L}} \rangle$ is the order at stage $s$. At each stage, one element of $\omega$ is enumerated between each pair of adjacent blocks and at each end of the sequence of blocks. New elements will never be enumerated into blocks. These newly enumerated elements will be singleton blocks.

Notice that if $\mathcal{S} = \langle S, <_{\mathcal{L}} \rangle$ is a suborder of $\mathcal{L}$ isomorphic to $\mathbb{Q}$, then $S$ can contain at most one element from each block. We leverage this fact to create $\mathcal{L}$ with the required properties. We build $\mathcal{L}$ to ensure that

1. $\mathcal{L}$ has a suborder $\mathcal{S} = \langle S, <_{\mathcal{L}} \rangle$ isomorphic to $\mathbb{Q}$,
2. for every such $\mathcal{S}$, $S$ is hyperimmune.

1 ensures that $\mathcal{L}$ is not scattered and 2 is sufficient to ensure that $\mathcal{L}$ is weakly scattered. To see this, let $\mathcal{S}$ be a suborder of $\mathcal{L}$ isomorphic to $\mathbb{Q}$ and consider $P = \mathrm{Seg}(\mathcal{S})$. Then $\mathrm{Brl}(P) = \{|\sigma| : \sigma \in \mathrm{Br}(P)\} = S$ is hyperimmune and so $P$ is small by the observation in Definition 3.2. Therefore $P$ does not contain a computably perfect subclass, and so $\mathcal{L}$ is weakly scattered.

During the construction to satisfy a requirement we may merge adjacent blocks into one block. This will change the order relation on the blocks but not on $\mathcal{L}$. Each block will thus be a finite c.e. set. $B_{i,s}$ denotes the set of elements enumerated into $B_i$ by stage $s$. Each block $B_{i,s}$ has associated with it a *restraint number* $r(i, s)$. No requirement may force the merging of a block unless the index of the requirement is lower than the restraint number of the block. All newly enumerated singleton blocks $B_{i,s}$ have $r(i, s) = \infty$. We will satisfy the following requirements.

$Q$: $\mathcal{L}$ contains a suborder isomorphic to $\mathbb{Q}$.

$R_e$: for any suborder $\mathcal{S}$ of $\mathcal{L}$ isomorphic to $\mathbb{Q}$, then $\{e\}$ does not dominate $p_S$ the principal function of $S$.

Requirement $Q$ will be satisfied by the enumeration at each stage of singleton blocks between and at each end of existing blocks, and we need to take no other action to ensure its satisfaction.

In order to satisfy requirement $R_e$, we wait for a stage $s$ at which $\{e\}_s(n)\downarrow$ for some $1 \leqslant n \leqslant s$. We then attempt to merge a sufficient number of adjacent blocks to ensure that $||\{B_{i,s+1} : i \leqslant \{e\}_s(n)\}|| \leqslant n$. If we succeed in this, then $S$ will contain at most $n$ elements less that or equal to $\{e\}(n)$ and so $p_S(n) > \{e\}(n)$. Thus $\{e\}$ will not dominate $p_S$. Merging other blocks to satisfy other requirements later on in the construction will never injure $R_e$ (as the merging of blocks can never decrease the principal function of $S$) and so the requirement is satisfied forever. We thus need to show only that each requirement has an opportunity to act.

We say requirement $R_e$ *requires attention at stage* $s$ if it is not satisfied (see below) and if there is an $n \leqslant s$ such that $\{e\}_s(n)\downarrow$ and it is possible to

1. enumerate all unused numbers up to and including $\{e\}(n)$ onto the right-hand end of $\mathcal{L}_s$, as singleton blocks,
2. and then merge some number of blocks (respecting the restraint numbers) so that $||\{B_{i,s+1} : i \leqslant \{e\}(n)\}|| \leqslant n$.

If $e$ is the lowest number such that $R_e$ requires attention, we take the following action: we carry out 1 and 2 above, and change the restraint numbers of any newly merged blocks to $e$. The requirement is then declared to be satisfied.

Lemma 4.6. *$\mathcal{L}$ contains a copy of $\mathbb{Q}$.*

Proof. Each requirement $R_e$ can act at most once, as once it acts it is declared satisfied and never again requires attention. No block can be merged by a requirement $R_e$ if its restraint number is less than $e$. Thus each block can be merged only a finite number of times, and is thus finite. Let $S$ be a subset of $L$ containing exactly one element from each block. The claim is that $\langle S, <_{\mathcal{L}}\rangle$ is isomorphic to $\mathbb{Q}$.

Let $a, b \in S$ such that $a <_{\mathcal{L}} b$. Let $s$ be a stage after which neither $B_a$ nor $B_b$ are merged with any other block. At stage $s+1$ elements $x, y, z$ are enumerated so that $B_x <_{\mathcal{L}} B_a <_{\mathcal{L}} B_y <_{\mathcal{L}} B_b <_{\mathcal{L}} B_z$. Thus $\langle S, <_{\mathcal{L}}\rangle$ is a countable dense linear order without endpoints and hence isomorphic to $\mathbb{Q}$. ⊣

Lemma 4.7. *If $S \subseteq L$ such that $\langle S, <_{\mathcal{L}}\rangle$ is isomorphic to $\mathbb{Q}$, then $S$ is hyperimmune.*

Proof. We show that if $S$ contains exactly one element from every block, then $S$ is hyperimmune. This is sufficient as any subset of $L$ that is isomorphic to $\mathbb{Q}$ contains at most one element from each block, and is thus a subset of such an $S$ and therefore also hyperimmune.

Let $\{e\}$ be any total computable function and let $s$ be a stage at which no requirement $R_i$ requires attention after stage $s$ for any $i < e$. Let $m = ||\{B_{i,s} : r(i,s) < e\}||$. No such $B_{i,s}$ will merge at any later stage, and thus $B_{i,s} = B_i$

for all $i < e$. We wait for a stage $t \geqslant s$ such that $\{e\}_t(2m+1)\downarrow$. We can then enumerate all unused numbers less than and equal to $\{e\}_t(2m+1)$ at the righthand end of the order and proceed to merge blocks as follows.

Let

$$B_{i_1,t} <_{\mathcal{L}} B_{i_2,t} <_{\mathcal{L}} \cdots <_{\mathcal{L}} B_{i_m,t}$$

be the set of blocks with $r(i,t) < e$. We then merge all blocks $B_{j,t} <_{\mathcal{L}} B_{i_1,t}$ into one block, and similarly with all blocks $B_{j,t} >_{\mathcal{L}} B_{i_m,t}$. We also, for each $k < m$, merge into one block all blocks $B_{j,t}$ with $B_{i_k,t} <_{\mathcal{L}} B_{j,t} <_{\mathcal{L}} B_{i_{k+1},t}$. We are thus left with $2m+1$ blocks in the ordering. All numbers less than or equal to $\{e\}(2m+1)$ are contained within these blocks so $p_S(2m+1) > \{e\}(2m+1)$ and $\{e\}$ does not dominate $p_S$. As $\{e\}$ was an arbitrary total computable function, $S$ is hyperimmune. ⊣

⊣

We are primarily concerned here with computable linear orders. If $\mathcal{L}$ is a computable linear ordering, then $\mathrm{Seg}(\mathcal{L})$ forms a $\Pi^0_1$ class by inspection of the above definition . Furthermore,

THEOREM 4.8. *If a computable linear order $\mathcal{L}$ is not weakly scattered, it contains a co-c.e. copy of $\mathbb{Q}$.*

PROOF. If $\mathrm{Seg}(\mathcal{L})$ contains a computably perfect subclass, it contains a computably perfect $\Pi^0_1$ subclass $P$ by Lemma 2.4. As above, the set $\mathrm{Brl}(P) = \{|\sigma| : \sigma \in \mathrm{Br}(P)\}$ is isomorphic to $\mathbb{Q}$. The branching level set of any $\Pi^0_1$ class is co-c.e. ⊣

THEOREM 4.9. *The following are equivalent for a computable linear order $\mathcal{L}$*

1. *$\mathcal{L}$ is not weakly scattered,*
2. *Seg($\mathcal{L}$) contains a complex element,*
3. *Seg($\mathcal{L}$) contains a wtt-cover for $2^\omega$.*

Corollary 4.9 in [8] states that if $0'$ is wtt-reducible to an element of $\mathrm{Seg}(\mathcal{L})$, then there is a $\Delta^0_2$ suborder isomormphic to $\mathbb{Q}$. The following improvement is to some degree inherent in their work:

COROLLARY 4.10. *If Seg($\mathcal{L}$) contains a complex element, then $\mathcal{L}$ has a co-c.e. suborder isomorphic to $\mathbb{Q}$. Furthermore, the converse holds if $\mathcal{L}$ is not weakly scattered.*

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