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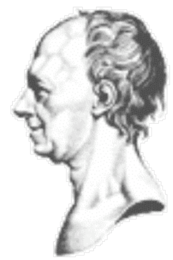


How Euler Did It

by Ed Sandifer

PDEs of fluids

September 2008

For his whole life Euler was interested in fluids and fluid mechanics, especially their applications to shipbuilding and navigation. He first wrote on fluid mechanics in his Paris Prize essay of 1727, [E4], *Meditationes super problemate nautico, de implantatione malorum …*, (Thoughts about a navigational problem on the placement of masts), an essay that earned the young Euler an *acessit*, roughly an honorable mention, from the Paris Academy. Euler's last book, *Théorie complete de la construction et de la manoeuvre des vaisseaux*, (Complete theory of the construction and maneuvering of ships) [E426], published in 1773, also dealt with practical applications of fluid mechanics. We could summarize Euler's contribution to the subject by saying that he extended the principles described by Archimedes in *On floating bodies* from statics to dynamics, using calculus and partial differential equations. Indeed, he made some of the first practical applications of partial differential equations.

Euler's work is very well known among people who study fluid mechanics. Several of the fundamental equations that describe non-turbulent fluid flow are known simply as "the Euler equations," and the problem of extending those equations to turbulent flow, the Navier-Stokes equations, is one of the great unsolved problems of our age.

This month, we are going to look at [E258], *Principia motus fluidorum,* (Principles of the motion of fluids), in which Euler derives the partial differential equations that describe two of the basic properties of fluids:

1. the differential equations for the continuity of incompressible fluids, and
2. the dynamical equations for ideal incompressible fluids.

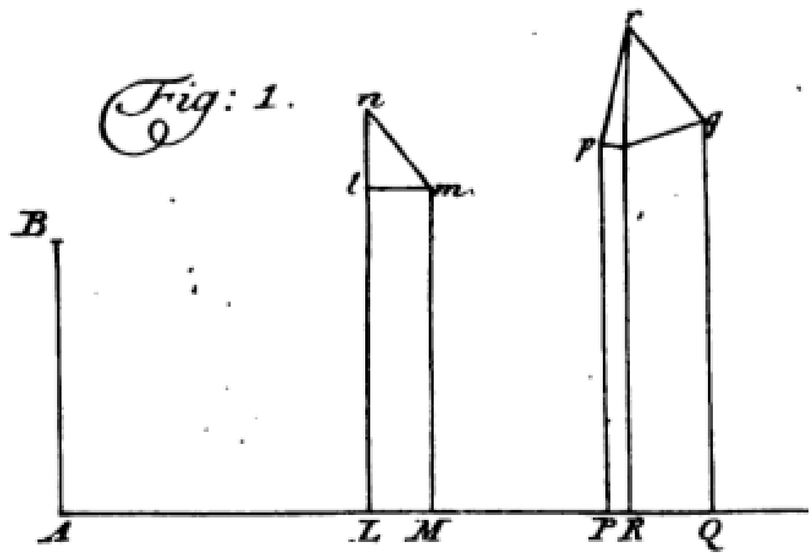
We will examine the first of these derivations in detail.

Euler begins by warning us how much more complicated fluids are than solids. If we know the motions of just three points of a rigid solid, then we can determine the motion of the entire body. For fluids, though, different parts of the fluid can have very different motions. Even knowing the flows of many points still leaves infinitely many possible flows.

This is not to say that there are no laws regarding fluid flow, and one of Euler's favorite is the Law of Impenetrability, that two objects cannot occupy the same space at the same time. This will be Euler's main tool in his analysis. He will describe what we would call "volume elements" and regard them as individual objects, which therefore must obey the Law of Impenetrability. Moreover, Euler will assume that the fluid flow is continuous and incompressible. Euler's notion of continuity was a bit different than ours is today, and his notion implied that the flow also be differentiable. This kind of flow is now called incompressible laminar flow. This rules out phenomena like breaking into droplets, forming cavities and flowing around obstacles. Thus Euler seeks to describe what he calls *possible* motions, those that are both incompressible and continuous.

For the first part of his paper, the part in which he derives the PDEs of incompressible fluids, he additionally assumes that the fluid body is subject to no forces or pressures. In the end, this last assumption does not change Euler's conclusions.

Euler begins with the two-dimensional case and asks us to consider an arbitrary point l in the fluid body. (See Fig. 1.) He takes his axes to be the lines AQ and AB, so the coordinates of the point l are $AL = x$ and $Ll = y$.



With respect to these axes, we resolve the motion of the point l into its two components, $u = lm$ parallel to the x-axis and $v = ln$ parallel to the y-axis. He notes that the speed of flow at point l is thus $\sqrt{uu + vv}$ and its direction, given as an angle relative to the x-axis is $\arctan\frac{u}{v}$.

Now, these velocity components u and v are not constant, but they vary with x and y. Euler introduces functions L, l, M and m so that he can describe these variations as differentials, writing

$$du = ldx + Ldy \quad \text{and}$$
$$dv = mdx + Mdy.$$

We would write $L = \frac{\partial u}{\partial x}$, $l = \frac{\partial u}{\partial y}$, $M = \frac{\partial v}{\partial x}$ and $m = \frac{\partial v}{\partial y}$. Both pairs, L and l, M and m, are themselves partial derivatives, so $\frac{\partial L}{\partial y} = \frac{\partial l}{\partial x}$ and $\frac{\partial M}{\partial y} = \frac{\partial m}{\partial x}$. When Euler wrote this paper in 1752, this fact that "mixed partial derivatives are equal" was still a fairly recent result. [E44, Sandifer 2004]

If a new point P (again, see Fig. 1) is located at distances dx and dy relative to the point l, then the velocity components at the point P will be

$$u + Ldx + ldy \quad \text{and}$$
$$v + Mdx + mdy.$$

Euler cleverly reuses his points *m* and *n* by letting *lmn* be a triangular element of water, and takes $lm = dx$ and $ln = dy$. He reassures us that,[1] "The whole mass of the fluid can be mentally divided up into elements like this, so that what we prescribe for one element will apply equally well to all." Euler seeks to describe the points *p, q* and *r* to which the points *l, m* and *n* respectively are moved by the flow during the time interval *dt*. He calls this time interval a *tempusculo*, or "tiny time interval." I don't recall seeing this word before, and I wonder if it is simply uncommon, if it was a bit of "math slang" popular in the 1750s, or if Euler constructed it just for the occasion.

To locate the points *p, q* and *r*, Euler begins by giving us table of the velocities in the *x* and *y* directions (i.e. parallel to AL and AB, respectively) at the points *l, m* and *n*:

point:	l	m	n
speed in the *x* direction	u	$u + Ldx$	$u + ldy$
speed in the *y* direction	v	$v + Mdx$	$v + mdy$

This lets him write the coordinates of each of the points *p, q* and *r*. First,

$$AP - AL = udt \quad \text{and}$$
$$Pp - Ll = vdt.$$

This gives the coordinates of *p* as $A = AL + udt$ and $Pp = Ll + vdt$. Similarly, the coordinates of *q* are $AQ = AM + (u + ldx)dt$ and $Qq = Mm + (v + Mdx)dt$, and those of *r* are $AR = AL + (u + ldy)dt$ and $Rr = Ln + (v + mdy)dt$.

Euler gives a brief argument that *pqr* is still triangle because triangle *lmn* and the time interval *dt* are infinitely small. Then, "[s]ince the element *lmn* ought not be extended into a greater area, nor to be compressed into a smaller one, its motion must be so composed that the area of triangle *pqr* equals the area of triangle *lmn*." This is a key observation, for it allows Euler to describe his notion of incompressibility by giving conditions that preserve the area of the element *lmn*. Euler does not yet have access to the Divergence theorem that many of us learn in our third semester calculus course, because vector fields and their accompanying notions of gradient, divergence and curl, did not arise until the 19th century. Hence Euler has to do all the work directly, "from scratch."

Triangle *lmn* is a right triangle, so it is easy to find its area, $\frac{1}{2}dy\,dx$.

To find the area of triangle *pqr,* we refer again to Fig. 1, and see that the area is the sum of the areas of trapezoids *PprR* and *RrqQ*, less the area of trapezoid *PpqQ*. In formulas,

$$PprR = \frac{1}{2}PR(Pp + Rr),$$
$$RrqQ = \frac{1}{2}RQ(Rr + Qq) \quad \text{and}$$
$$PpqQ = \frac{1}{2}PQ(Pp + Qq).$$

[1] Here and elsewhere in this column, when I quote Euler's words, I use the translation graciously provided by Enlin Pan.

Substituting and collecting terms gives

$$pqr = \tfrac{1}{2} PQ \cdot Rr - \tfrac{1}{2} RQ \cdot Pp - \tfrac{1}{2} PR \cdot Qq.$$

Now, let $PQ=Q$, $PR=R$ so that $RQ=Q-R$. Also let $Qq=Pp+q$ and $Rr=Pp+r$, and the area formula simplifies to

$$pqr = \tfrac{1}{2} Q \cdot r - \tfrac{1}{2} R \cdot q.$$

From what came earlier, we can rewrite the elements of the right hand side using differentials to get

$$Q = dx + Ldxdt, \qquad q = mdxdt,$$
$$R = ldydt \quad \text{and} \qquad r = dy + mdydt.$$

Now, the details are interesting, so we'll include them.

Substitution, then factoring gives

$$pqr = \tfrac{1}{2} dxdy\left(1 + Ldt\right)\left(1 + Mdt\right) - \tfrac{1}{2} Mldxdydt^2,$$

which combines to give

$$pqr = \tfrac{1}{2} dxdy\left(1 + Ldt + mdt + Lmdt^2 - Mldt^2\right).$$

Because we know that $\Delta pqr = \tfrac{1}{2} dxdy$ and that $dxdy \neq 0$, we substitute, then subtract and cancel to get

$$Ldt + mdt + Lmdt^2 - Mldt^2 = 0$$

or

$$L + m + Lmdt - Mldt = 0.$$

As dt vanishes, this gives

$$L + m = 0.$$

In terms of partial derivatives, Euler writes

$$\frac{du}{dx} + \frac{dv}{dy} = 0,$$

but we would write

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$

or even

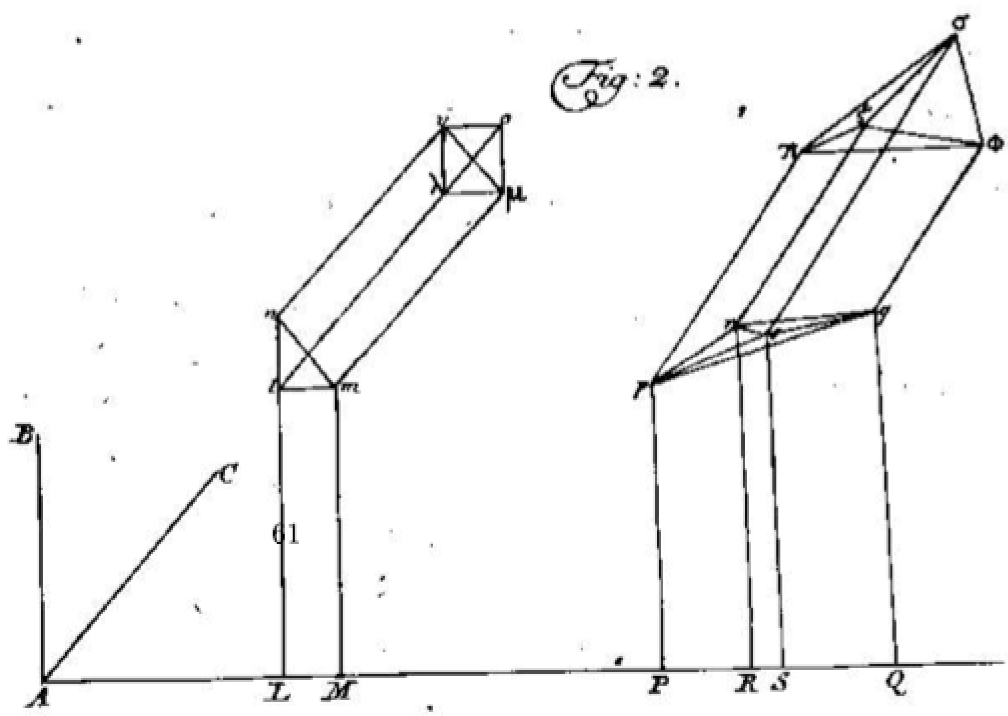
$$\nabla \cdot \mathbf{u} = 0.$$

This last notation was at least 100 years in Euler's future.

An analogous argument, based on Fig. 2, seven pages long instead of four, leads to the analogous conclusion for three dimensions, which Euler writes

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0.$$

It is interesting to note that, except for the relative positions of points *p, q* and *r*, Fig. 1 is a proper subset of Fig. 2. Likewise, the argument leading to Euler's two dimensional formula is a proper subset of the analogous three dimensional argument as well.



At this point, we've described only the first part of Euler's paper. In the first part, Euler had not concerned himself with how the flow might change over time. The key new idea in the second part is that forces, internal and external, might cause the flow itself to change with time. Thus, his equations of motion in two dimensions have an extra term to describe how the flow changes with time. In particular, and in modern terms, he begins with

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial t}dt \quad \text{and}$$
$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy + \frac{\partial v}{\partial t}dt.$$

This is clearly analogous to his starting point in the first section, where his first equations were

$$du = ldx + Ldy \quad \text{and}$$
$$dv = mdx + Mdy.$$

In fluids, change in velocity is related to pressure, denoted by p. As in the first part of the paper, Euler uses his initial equations to derive the differential equations, first in two dimensions, then in three, that describe the pressures in fluid flow. For three dimensions, and in modern terms, they are

$$\frac{\partial p}{\partial x} = -2\left(\frac{\partial u}{\partial x}u + \frac{\partial u}{\partial y}v + \frac{\partial u}{\partial z}w + \frac{\partial u}{\partial t}\right),$$
$$\frac{\partial p}{\partial y} = -2\left(\frac{\partial v}{\partial x}u + \frac{\partial v}{\partial y}v + \frac{\partial v}{\partial z}w + \frac{\partial v}{\partial t}\right) \quad \text{and}$$
$$\frac{\partial p}{\partial z} = -1 - 2\left(\frac{\partial w}{\partial x}u + \frac{\partial w}{\partial y}v + \frac{\partial w}{\partial z}w + \frac{\partial w}{\partial t}\right).$$

In the interests of brevity, we will not give more details of these derivations. The interested reader is encouraged to consult [Truesdell 1954] and Enlin Pan, in his English translation of [E258].

Finally, sharp-eyed readers who read the references first may note three other articles that Euler wrote about fluids, [E225], [E226] and [E227]. From their titles "General principles on the state of fluid equilibrium", "General principles on the movement of fluids" and "Continuation of the researches on the theory of the movement of fluids", they seem to cover much of the same material as [E258]. From their length, a total of 131 pages, compared to 36 pages for E258, they seem to cover the material in more depth, and from their Eneström numbers, all less than 258, it is evident that they were all published before E258. Why, then, did we describe E258 instead of E225, E226 and E227?

E258 was actually written first, in 1752, but did not appear in the journals of the St. Petersburg Academy until its volume for papers presented in the years 1756 and 1757. Typical publication delays delayed the actual printing of that volume until 1761. The other three were written in 1753, 1755 and 1755 respectively in French for the journal of the Berlin Academy and published in that academy's volume for the year 1755, printed in 1757. Euler's colleagues in Berlin would have learned of the results in 1753 and 1755, and scientists elsewhere would have learned them in 1757. Hence, if we are studying the influence of Euler's ideas, then we should have looked at the three papers in French. Moreover, they represent a more refined and complete treatise on the subject.

However, if we wish to see the growth of Euler's ideas, to see "how Euler did it," then we should look at his earlier work on the subject, and that is E258. Readers interested in the differences between E258 and the three treatises in French should consult [Truesdell 1954]. Truesdell's "Introduction" itself is an outstanding essay, and it's in English!

Special thanks to Enlin Pan for allowing me to use his English translation of E258. It was most helpful.

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