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How Euler Did It

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A false logarithm series

December 2007

Solving a good research question should open more doors than it closes. One of Euler's lesser papers, *Methodus generalis summandi progressiones* ("General methods of summing progressions") [E25] is more noteworthy for the things it started than the things it finished. The principal role of the paper is as one of a sequence of papers that led to Euler's development of the Euler-Maclaurin summation formula. That sequence began in 1729 with a letter to Goldbach containing results that Euler later published in 1738 in [E20], and continued through [E25], [E43], [E46], [E47] up to [E55], *Methodus universalis series summandi ulterius promota,* written in 1736 and published in 1741.

This sequence of papers has a wonderful plot. First Euler examines the relations between the partial sums of the harmonic series, $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n}$ and the logarithm function, ln n. Then he sharpens these relations by using the fact that the function $1/x$ "naturally expresses" the terms of the harmonic series and that $\int_1^n \frac{1}{x}\,dx = \ln n$. He extends his results first to other more general series of reciprocals like $1+\frac{1}{2^k}+\frac{1}{3^k}+\frac{1}{4^k}+\cdots+\frac{1}{n^k}$, then between partial sums of other series of reciprocals and their corresponding integrals $\int_1^n \frac{1}{x^k}\,dx$, and then to functions in general, $\sum_{i=1}^{n} f(i)$ and $\int_1^n f(x)\,dx$. It is delightful to watch the young Euler sharpen his tools and his insights from one year to the next between the years 1729 and 1736.

At the end of E25, though, Euler casts a glance in another direction, and poses two series for which the methods he had used to evaluate so many other series do not seem to work:

$$1+\frac{1}{3}+\frac{1}{7}+\frac{1}{15}+\cdots+\frac{1}{2^n-1}+\text{etc. and } \frac{1}{3}+\frac{1}{7}+\frac{1}{8}+\frac{1}{15}+\frac{1}{24}+\frac{1}{27}+\text{etc.}$$

The pattern for the second of these series is not obvious, but Euler explains that "the general term is $\frac{1}{a^a-1}$ where a and a denote integers greater than one." Goldbach had shown that this series sums to

1, and Euler expanded on Goldbach's technique in [E72], one of Euler's greatest papers, to discover the Euler product formula. See [S, BPV, D].

This month's topic, though is the first of these series. It comes up again almost 15 years later in [E190], *Consideratio quarumdam serierum quae singularibus proprietatibus sunt praeditae* ("Consideration of some series which are distinguished by special properties"). In E190, Euler studies the series

$$s=\frac{1-x}{1-a}+\frac{(1-x)(a-x)}{a-a^3}+\frac{(1-x)(a-x)(a^2-x)}{a^3-a^6}+\frac{(1-x)(a-x)(a^2-x)(a^3-x)}{a^6-a^{10}}\text{ etc.}$$

Here, the numerators add a factor of $a^n - x$ at each term, and the denominators involve exponents that are triangular numbers. In modern notation, one might write

$$s=\sum_{n=0}^{\infty}\frac{\prod_{k=0}^{n}\left(a^k-x\right)}{a^{T(n)}-a^{T(n+1)}}$$

where $T(n)$ is the nth triangular number, given by $T(n) = n\cdot(n + 1)/2$.

It takes some thought to recognize what "special properties" might distinguish this series, even if we follow 18th century style and ignore questions of convergence. Adventurous readers may want to play with the series for a few minutes before reading on.

Note that if $x = a$, then the first term equals 1, and all the rest of the terms are zero, since they have a factor of $a - x$ in their numerators. Hence $s(a)=1$.

Further, if $x=a^2$ then the first term reduces to $1+a$ and the second term reduces to $1-a$, while all the other terms vanish, so $s\left(a^2\right)=2$. Similarly, Euler observes that $s\left(a^n\right)=n$ for positive whole numbers n, though Euler only shows us the calculations up to $n = 3$ and claims to have done them himself up to $n = 5$. Later in the article, Euler gives a proof that $s\left(a^n\right)=n$ when n is a positive integer, but at this point he gives only evidence. This evidence, though, naturally leads to the conjecture that $s(x)=\log_a x$.

But it isn't. Euler demonstrates this by showing that, for $a = 10$, $s(9) = 0.89705\ 05852\ 10673\ 21224$ but $\log 9 = 0.95424\ 2509$ (though the editors of the *Opera omnia* note that Euler made an error in his calculation of $s(9)$. It should be $0.89777\ 85865\ 88$. It still isn't log 9.) It is entertaining to check this using your favorite computer algebra system.

Euler apparently picked $a = 10$ and $x = 9$ rather charitably. In fact your computer algebra system can show that most of the time, $s(x)$ and $\log_a x$ are not very close together at all. Euler did not have a computer algebra system. He does have an easier way, not involving approximations, to show that $s(x)$ is not a logarithm function. He takes $x = 0$, and he "knows" that $\log 0 = -\infty$. On the other hand, for $x = 0$, his series gives

$$\frac{1}{1-a}+\frac{1}{1-aa}+\frac{1}{1-a^3}+\frac{1}{1-a^4}+\frac{1}{1-a^5}+\text{etc.}$$

This series has a finite sum, so $s(x)$ cannot be the logarithm function.

Note, though, that this series is the negative of one of the series that Euler proposed at the end of E25, and here again he tells us, "this series cannot be summed."

It must have both disappointed and excited Euler that $s(x)$ was not the logarithm function. On the one hand, if the series *were* the logarithm function, then it would have provided an unusually fast-converging means of calculating logarithms. On the other hand, since $s(x)$ is *not* the logarithm function, it challenged one of his basic assumptions. He had two functions, $s(x)$ and $\log_a x$ that "naturally expressed" the same sequence of numbers; that is, they agreed at infinitely many values of x, yet they were not the same function.

This is in section 4 of this 32-section article. Euler spends most of the rest of this article studying properties of his series, showing how much it really does differ from the logarithm functions, and also showing rigorously that it does agree with the logarithm function at integer powers of a. We'll omit that, and refer the interested reader to the Mattmueller translation available on The Euler Archive. [E190]

Instead, we will leap forward to section 28, where Euler returns to the series from E25,

$$\frac{1}{a-1}+\frac{1}{a^2-1}+\frac{1}{a^3-1}+\frac{1}{a^4-1}+\frac{1}{a^5-1}+\text{etc.}$$

Of this, Euler writes (in the Mattmueller translation), "… for $a > 1$, even though it is finite and can easily be determined by approximations, [it] cannot be expressed neither in rational nor in irrational numbers. It appears therefore especially worth the effort that mathematicians investigate the nature of that transcendental quantity by which its sum is expressed." Here, Euler calls "irrational" what we would call "algebraic", though he uses the word "transcendental" in its modern sense.

Unable to express the series exactly, he sets out to give good approximations. He defines a new series s, not the same one he also denoted by s earlier in this article, as

$$s=\frac{1}{a-z}+\frac{1}{a^2-z}+\frac{1}{a^3-z}+\frac{1}{a^4-z}+\frac{1}{a^5-z}+\text{etc.}$$

Then the series Euler wants to approximate is the value of this series s when $z = 1$. He sets to work on this new series. Euler skips a few steps here that we will put in. The first term of this series can be expanded into a geometric series as

$$\frac{1}{a-z}=\frac{1/a}{1-z/a}=\frac{1}{a}+\frac{z}{a^2}+\frac{z^2}{a^3}+\frac{z^3}{a^4}+\text{etc.}$$

Likewise, the second term expands as

$$\frac{1}{a^2 - z} = \frac{1/a^2}{1 - z/a^2} = \frac{1}{a^2} + \frac{z}{a^4} + \frac{z^2}{a^6} + \frac{z^3}{a^8} + \text{etc.}$$

The other terms expand similarly. Euler "reverses the order of summation" to gather together like powers of z, and gives us s in a different form as

$$\begin{aligned} s &= \frac{1}{a} + \frac{1}{a^2} + \frac{1}{a^3} + \frac{1}{a^4} + \frac{1}{a^5} + \text{etc.} \\ &+ z\left(\frac{1}{a^2} + \frac{1}{a^4} + \frac{1}{a^6} + \frac{1}{a^8} + \frac{1}{a^{10}} + \text{etc.} \right) \\ &+ z^2\left(\frac{1}{a^3} + \frac{1}{a^6} + \frac{1}{a^9} + \frac{1}{a^{12}} + \frac{1}{a^{15}} + \text{etc.} \right) + \text{etc.} \end{aligned}$$

This done, Euler uses a trick that he had also used in E25. He knows that most of the error in the approximation of the sum of a series occurs in the first few terms of the series. To reduce this effect, he takes A to be the sum of the first n terms of s; that is

$$A = \frac{1}{a - z} + \frac{1}{a^2 - z} + \frac{1}{a^3 - z} + \frac{1}{a^4 - z} + \cdots + \frac{1}{a^n - z}.$$

This leaves

$$s = A + \frac{1}{a^{n+1} - z} + \frac{1}{a^{n+2} - z} + \frac{1}{a^{n+3} - z} + + \frac{1}{a^{n+4} - z} + \text{etc.}$$

Euler again expands this into geometric series and collects like terms to get

$$\begin{aligned} s &= A + \frac{1}{a^{n+1}} + \frac{1}{a^{n+2}} + \frac{1}{a^{n+3}} + \frac{1}{a^{n+4}} + \text{etc.} \\ &+ z\left(\frac{1}{a^{2n+2}} + \frac{1}{a^{2n+4}} + \frac{1}{a^{2n+6}} + \frac{1}{a^{2n+8}} + \text{etc.} \right) \\ &+ z^2\left(\frac{1}{a^{3n+3}} + \frac{1}{a^{3n+6}} + \frac{1}{a^{3n+9}} + \frac{1}{a^{3n+12}} + \text{etc.} \right) + \text{etc.} \end{aligned}$$

Each line in this expression is a geometric series, so he can sum those to get

$$s = A + \frac{1}{a^n (a - 1)} + \frac{z}{a^{2n} (aa - 1)} + \frac{z^2}{a^{3n} (a^3 - 1)} + \frac{z^3}{a^{4n} (a^4 - 1)} + \text{etc.}$$

For the series Euler proposed in E25, the case $a = 2$ and $z = 1$, this gives

$$s = A + \frac{1}{1 \cdot 2^n} + \frac{1}{3 \cdot 2^{2n}} + \frac{1}{7 \cdot 2^{3n}} + \frac{1}{15 \cdot 2^{4n}} + \frac{1}{31 \cdot 2^{5n}} + \text{etc.}$$

He takes $n = 4$ and so sums the first four terms of the series to get

$$A = 1 + \frac{1}{3} + \frac{1}{7} + \frac{1}{15} = 1.542857142857141\,.$$

This is incorrect in the last decimal place, which should be a 3. That makes

$$s = A + \frac{1}{16 \cdot 1} + \frac{1}{16^2 \cdot 3} + \frac{1}{16^3 \cdot 7} + \frac{1}{16^4 \cdot 15} + \text{etc.}$$

Euler sums the first 15 terms of this series to get

$$\begin{aligned} s &= A + 0.063638009558149 \\ &= 1.606695152415291 \end{aligned}$$

This agrees exactly with my computer algebra system for the infinite series.

This is the best Euler can do with that series from E25. He has one last remark, though. If we go back to the series from the beginning of section 28,

$$\frac{1}{a-1} + \frac{1}{a^2-1} + \frac{1}{a^3-1} + \frac{1}{a^4-1} + \frac{1}{a^5-1} + \text{etc.}$$

and if we expand each of the terms as geometric series, then collect like powers of a, we get the form

$$s = \frac{1}{a} + \frac{2}{a^2} + \frac{2}{a^3} + \frac{3}{a^4} + \frac{2}{a^5} + \frac{4}{a^6} + \frac{2}{a^7} + \frac{4}{a^8} + \frac{3}{a^9} + \text{etc.},$$

where the nth numerator counts the number of divisors of n. Euler does not try to explain why this is true (though it is), but he does tell us that the numerator in the term $\frac{4}{a^6}$ is a 4 because the exponent 6 has four divisors, namely 1, 2, 3 and 6. For prime exponents, the numerator will always be 2, and for composite exponents it will always be greater than 2. These numerators are easy to calculate, and for the special case $a = 10$, it is easy for us to sum the series.

$$s = \frac{1}{9} + \frac{1}{99} + \frac{1}{999} + \frac{1}{9999} + \frac{1}{99999} + \text{etc.}$$

Euler gives the sum to 30 decimal places:

$$s = 0.122324243426244526264428344628\,.$$

It is clear that the series has number theoretic properties, but Euler did not pursue them any farther. Those properties are related to what we now call q-series. They were extensively studied by such great mathematicians as Gauss, Cauchy, Jacobi, Sylvester and Ramanujan and are still of great interest today.

I'd like to thank Warren Johnson, now at Connecticut College, for bringing this article to my attention, for helping me understand its connections with q-series, and for helpful comments on the text itself.

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