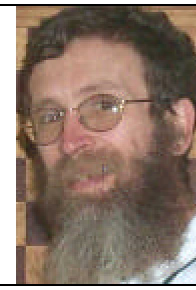
How Euler Did It

by Ed Sandifer



Introduction to Complex Variables

May 2007

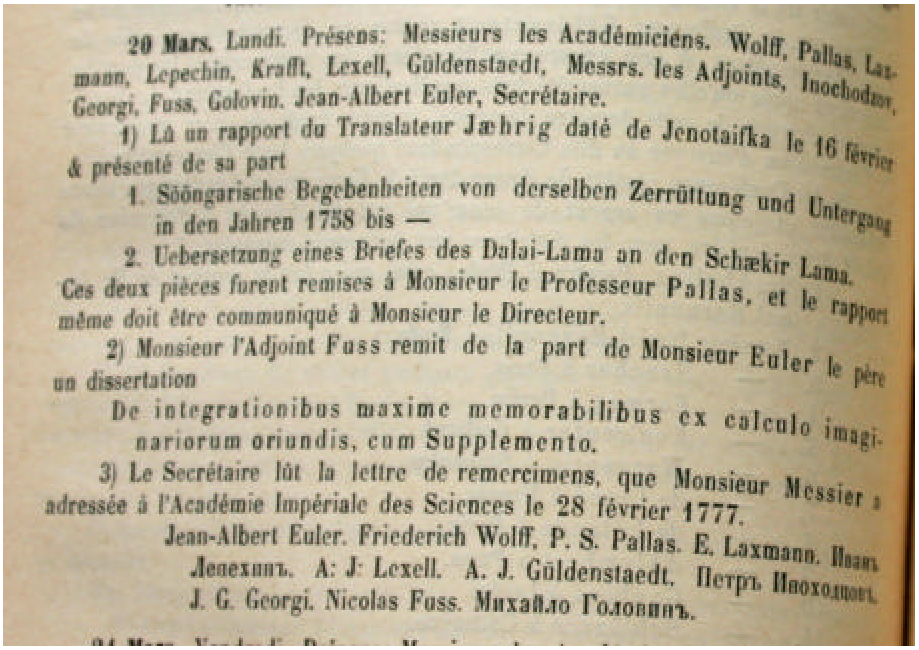
On Monday, March 20, 1777 the Imperial Academy of Sciences of St. Petersburg had one of its regular meetings. Except for holidays and occasional special meetings, they met twice a week on Mondays and Fridays, a total of 70 or 80 meetings per year.

This particular meeting wasn't much different from the other meetings they had that year, though it was a little shorter than most. The minutes from that meeting are shown in the photograph below. [SPA] It opened with a report from the Academy's translator, a Mr. Jaehrig, including his account of some letters between the Dalai Lama and the Sakya Trizin Lama, and it closed with the reading of a letter of thanks from someone named Monsieur Messier. In between, one of the Adjoint Members of the Academy, Nicolas Fuss, submitted on behalf of Leonhard Euler two articles, De integrationibus maxime memorabilibus ex calculo imaginariorum oriundis, (E656, "On some most memorable integrations arising from the calculus of the imaginaries") and its sequel, E657, Supplementum ad dissertationem praecedentem circa integrationem formulae $\int \frac{z^{m-1}dz}{1-z^n}$ casu, quo ponitur $z = v\left(\cos j + \sqrt{-1}\sin j\right)$, "Supplement to the preceding article about the integration of the formula $\int \frac{z^{m-1}dz}{1-z^n}$ by setting $z = v\left(\cos j + \sqrt{-1}\sin j\right)$"). Eleven days later, on March 31, Fuss brought five more articles, including one related to these two, Ulterior disquisitio de formulis integralibus imaginaries, (E694, "Later article on imaginary integral formulas").

The Academy's most famous member, Leonhard Euler, blind for more than five years, would turn 70 years old in just a month and seldom attended the regular meetings any more. Instead, he stayed at his home a few blocks from the Neva, the river through St. Petersburg, and his assistants went there to work with him. His assistants would do the actual writing, and they would sometimes work out the details of the calculations, but the articles were almost always published under Euler's name.

In 1777, Euler and his assistants sent more than 50 articles to the Academy. They would be "presented" to the Academy, that is, the manuscript was handed over to the Secretary of the Academy. Euler and company wrote articles too fast for the Academy's publishers, but those articles that were to be published without delay also had to be read aloud at a meeting of the Academy. These two articles weren't published until the 1789 issue of the Academy's journal, and that issue wasn't actually printed until 1793, sixteen years after it was written, and these articles apparently escaped being read before the Academy.

So, what are these "most memorable integrations" of which Euler writes? The main point of the articles is to show that calculus with complex numbers is possible, and that it works a lot like calculus with real numbers.



20 Mars. Lundi. Présens: Messieurs les Académiciens. Wolff, Pallas, Laxmann, Lepechin, Krafft, Lexell, Güldenstaedt. Messrs. les Adjoints, Inochodzow, Georgi, Fuss, Golovin. Jean-Albert Euler, Secrétaire.

1) Lû un rapport du Translateur Jæhrig daté de Jenotaifka le 16 février & présenté de sa part

1. Söongarische Begebenheiten von derselben Zerrüttung und Untergang in den Jahren 1758 bis —
2. Uebersetzung eines Briefes des Dalai-Lama an den Schækir Lama.

Ces deux pièces furent remises à Monsieur le Professeur Pallas, et le rapport même doit être communiqué à Monsieur le Directeur.

2) Monsieur l'Adjoint Fuss remit de la part de Monsieur Euler le père un dissertation

De integrationibus maxime memorabilibus ex calculo imaginariorum oriundis, cum Supplemento.

3) Le Secrétaire lût la lettre de remercimens, que Monsieur Messier a adressée à l'Académie Impériale des Sciences le 28 février 1777.

Jean-Albert Euler. Friederich Wolff. P. S. Pallas. E. Laxmann. Иванъ Лепехинъ. A: J: Lexell. A. J. Güldenstaedt. Петръ Иноходцовъ. J. G. Georgi. Nicolas Fuss. Михайло Головинъ.

Minutes of the Petersburg Academy from March 20, 1777

Euler begins asking us to consider a differential Zdz, where Z is a function of what he calls an "imaginary" quantity z. He writes its integral as $\int Zdz = \Delta : z$, where $\Delta : z$ is Euler's function notation. We would write $f(z)$ or $\Delta(z)$.

Now we separate everything in sight into real and imaginary parts. We take $z = x + y\sqrt{-1}$. (Euler and his students have not yet adopted the symbol i to denote $\sqrt{-1}$. They do that later in 1777.) Also, $Z = M + N\sqrt{-1}$ and $\Delta : z = P + Q\sqrt{-1}$.

Euler is very patient with us here, and explains that

$$Zdz = \left(dx + dy\sqrt{-1}\right)\left(M + N\sqrt{-1}\right)$$

and that the real and the imaginary parts (he uses those words) are $Mdx - Ndy$ and $(Ndx + Mdy)\sqrt{-1}$ respectively, and that $P = \int (Mdx - Ndy)$ and $Q = \int (Ndx + Mdy)$.

Euler's patience lapses for a moment here when he just tells us, without giving details, that "because of the integrability criteria" it follows that

$$\frac{\partial M}{\partial y} = -\frac{\partial N}{\partial x} \text{ and } \frac{\partial N}{\partial y} = \frac{\partial M}{\partial x}.$$

In fact, it is easy to derive these formulas. We need only know that the mixed partial derivatives of Z have to be equal, but it isn't as easy as the last few steps have been. He does this step in more detail eleven days later in his third article, E694.

These are, of course, the Cauchy-Rieman equations, used to such great effect two or three mathematical generations later by Augustin-Louis Cauchy (1789-1857) and Bernhard Riemann (1826-1866).

Now, Euler wants to show that certain calculus facts familiar for ordinary functions of real numbers are also true for complex numbers. He begins with $\int z^n dz = \frac{z^{n+1}}{n+1}$. As was customary at the time, Euler neglects the constant of integration unless he needs it.

In z^n, Euler substitutes $z = x + y\sqrt{-1}$, then expands the resulting binomial as

$$\left(x+y\sqrt{-1}\right)^n = x^n + \left(\frac{n}{1}\right)x^{n-1}y\sqrt{-1} - \left(\frac{n}{2}\right)x^{n-2}yy - \left(\frac{n}{3}\right)x^{n-3}y^3\sqrt{-1} + \text{etc.}$$

Euler and his students had only recently started writing the binomial coefficients as $\left(\frac{n}{k}\right)$. Sometime over the next few decades, people started to omit the fraction bar, leaving us with the modern notation, $\binom{n}{k}$. We will use Euler's notation. Separating his expanded binomial into M, its real part, and N, the imaginary part, he gets

$$M = x^n - \left(\frac{n}{2}\right)x^{n-2}yy + \left(\frac{n}{4}\right)x^{n-4}y^4 - \left(\frac{n}{6}\right)x^{n-6}y^6 + \text{etc.}$$

and

$$N = \left(\frac{n}{1}\right)x^{n-1}y - \left(\frac{n}{3}\right)x^{n-3}y^3 + \left(\frac{n}{5}\right)x^{n-5}y^5 - \text{etc.}$$

He does a similar substitution and expansion with the right hand side, $\frac{z^{n+1}}{n+1}$, and separates it into its real and imaginary parts, P and Q, then multiplies by $n + 1$, to get what he thinks P and Q *should* be, if his integral formula is correct:

$$(n+1)P = x^{n+1} - \left(\frac{n+1}{2}\right)x^{n-1}yy + \left(\frac{n+1}{4}\right)x^{n-3}y^4 - \left(\frac{n+1}{6}\right)x^{n-5}y^6 + \text{etc.}$$

and

$$(n+1)Q = \left(\frac{n+1}{1}\right)x^n y - \left(\frac{n+1}{3}\right)x^{n-2}y^3 + \left(\frac{n+1}{5}\right)x^{n-4}y^5 - \text{etc.}$$

On the other hand, he knows that P *has to be* given by $P = \int (Mdx - Ndy)$. Substituting his expressions for M and N gives the rather formidable expression:

$$P = \int \left\{ \begin{array}{l} dx\left(x^n - \left(\dfrac{n}{2}\right) x^{n-2} y^2 + \left(\dfrac{n}{4}\right) x^{n-4} y^4 - \left(\dfrac{n}{6}\right) x^{n-6} y^6 + \text{etc.} \right) \\ -dy\left(\left(\dfrac{n}{1}\right) x^{n-1} y - \left(\dfrac{n}{3}\right) x^{n-3} y^3 + \left(\dfrac{n}{5}\right) x^{n-5} y^5 - \text{etc.} \right) \end{array} \right\}$$

Euler sets out to show that these two expressions for P are equal. He says that he will do this by integrating "by parts," but he really means that he will integrate parts of the actual value of P given in the integral, and then show that they equal the corresponding parts of the expression $(n + 1)P$.

For the term that has x^{n+1} in its integral, he gets

$$\int x^n dx = \frac{x^{n+1}}{n+1},$$

and that agrees with the corresponding part of $(n + 1)P$.

For the terms that have x^{n-1} in their integral, he gets

$$-\int \left(\frac{n}{2}\right) x^{n-2} y^2 dx - \int \left(\frac{n}{1}\right) x^{n-1} y dy$$

and shows that these integrate to give their corresponding part of $(n + 1)P$ as well. He continues, pairing $\int \left(\frac{n}{4}\right) x^{n-4} y^4 dx$ with $\int \left(\frac{n}{3}\right) x^{n-3} y^3 dy$, and then stops, saying that the pattern is clear. He also omits the details in showing that the expression that follows from $Q = \int (Ndx + Mdy)$ agrees with the expression given as $(n + 1)Q$.

Euler agrees that that was hard work, and offers us an easier way. It is slightly less general, since the binomial series expansions can be made to work even if n is not an integer and his easier way only works when n is a positive integer. But it is considerably shorter and a good deal more elegant. From x and y, Euler creates two new variables, $v = \sqrt{xx + yy}$ and an angle j chosen so that, as Euler writes it, "$\text{tang.}\, j = \frac{y}{x}$." This makes

$$x = v\cos j \quad \text{and} \quad y = v\sin j .$$

Their differentials are

$$dx = dv\cos j - vdj \sin j \quad \text{and} \quad dy = dv\sin j + vdj \cos j .$$

Now Euler can use deMoivre's formula to get

$$\left(x + y\sqrt{-1}\right)^n = v^n \left(\cos nj + \sqrt{-1}\sin nj\right).$$

This has real and imaginary parts M and N respectively equal to

$$M = v^n \cos nj$$

and

$$N = v^n \sin nj .$$

A similar calculation shows that, for $\int z^n dz = \frac{z^{n+1}}{n+1}$, we would have to have

$$P=\frac{v^{n+1}\cos(n+1)j}{n+1} \text{ and } Q=\frac{v^{n+1}\sin(n+1)j}{n+1}.$$

On the other hand, we know that $P=\int (Mdx-Ndy)$ and $Q=\int (Ndx+Mdy)$. Euler skips a few steps in his substitution. We'll skip some, too, but not as many as Euler did. For P, we get

$$\begin{aligned}P&=\int (Mdx-Ndy)\\&=\int \left(\left(v^n\cos nj\right)\left(dv\cos j-vdj\sin j\right)-\left(v^n\sin nj\right)\left(dv\sin j+vdj\cos j\right)\right)\end{aligned}$$

Now, after a careful expansion and application of the trigonometric identities

$$\begin{aligned}\cos(n+1)j&=\cos(j+nj)\\&=\cos j\cos nj-\sin j\sin nj\end{aligned}$$

and

$$\begin{aligned}\sin(n+1)j&=\sin(j+nj)\\&=\sin j\cos nj+\cos j\sin nj\end{aligned}$$

we get that

$$P=\int \frac{v^{n+1}}{n+1}\cos(n+1)j \text{ and } Q=\frac{v^{n+1}\sin(n+1)j}{n+1},$$

as promised.

Indeed, Euler's second solution took him only a page of calculations to find both P and Q, whereas his first method had taken two and a half pages to find only the first three terms of P. Things aren't quite as rosy as he would have us believe, though, because he does skip a good number of easy but paper-consuming calculations.

We have described only the first five of the 44 pages of Euler's first paper. The second paper adds 17 pages, and the third another 18. Euler goes on (and on and on) to apply the same methods to integrate $\frac{dz}{1+zz}$ and get, as we would expect, $\tan^{-1}z$, and $\frac{dz}{1+z}$. He goes on to integrate the more general form $\frac{z^{m-1}dz}{1+z^n}$, which contains the last two as special cases.

Gradually, he comes to appreciate the power and convenience of the substitution $z=v\left(\cos j+\sqrt{-1}\sin j\right)$, and devotes the second paper to that substitution. The third paper gives Euler's derivation of the Cauchy-Riemann formulas in more detail, and then attacks some more general integrals like $\int \frac{z^{m-1}dz}{\left(a\pm bz^n\right)^l}$.

Over the next few months, Euler expands his use of complex numbers in calculus, and in a paper [E671] presented to the Academy on May 5, 1777, as he is studying the integral of $\frac{dj\cos j}{\sqrt[n]{\cos nj}}$, he writes that he will be using imaginary numbers and that "I will use i to denote $\sqrt{-1}$." His notation caught on.

On a technical level, we've seen exciting developments here. We see Euler discovering the Cauchy-Riemann formulas more than a decade before Cauchy was even born, and almost 50 years before Riemann, and we've done calculus with complex numbers.

Something has happened on a philosophical level as well. For most of his life, Euler was content to use a principle that Leibniz had called the Principle of Continuation. This said, roughly, that similar things ought to behave similarly. This gave Euler reason to use the same rules of calculation with infinite and infinitesimal numbers that he used for finite numbers and to treat solid bodies as if they were point masses. The Principle of Continuation *should* have allowed Euler to integrate complex functions just like he integrated real ones.

We can only speculate why Euler chose to try to be analytically rigorous when writing about complex variables. Perhaps he wrote this paper to explain the use of complex numbers to his students, especially to Nikolas Fuss. That could also explain why Euler was sometimes very careful about including details, so that his students would understand, but at other times he skipped them, to leave gaps for his students to fill in.

On the other hand, perhaps he or the people he was writing for did not agree that complex numbers, or, as he called them, *imaginary* numbers, are similar to real numbers, so he was not comfortable applying the Principle of Continuation.

We won't even worry about why Cauchy and Riemann got their names on those equations instead of Euler.

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Ed Sandifer (SandiferE@wcsu.edu) is Professor of Mathematics at Western Connecticut State University in Danbury, CT. He is an avid marathon runner, with 35 Boston Marathons on his shoes, and he is Secretary of The Euler Society (www.EulerSociety.org). His new book, *The Early Mathematics of Leonhard Euler*, was published by the MAA in December 2006, as part of the celebrations of Euler's tercentennial in 2007. The MAA will be publishing a collection of the *How Euler Did It* columns during the summer of 2007.
