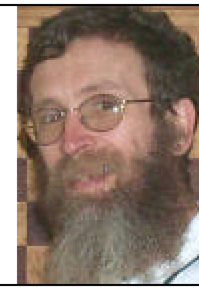
# How Euler Did It

by Ed Sandifer

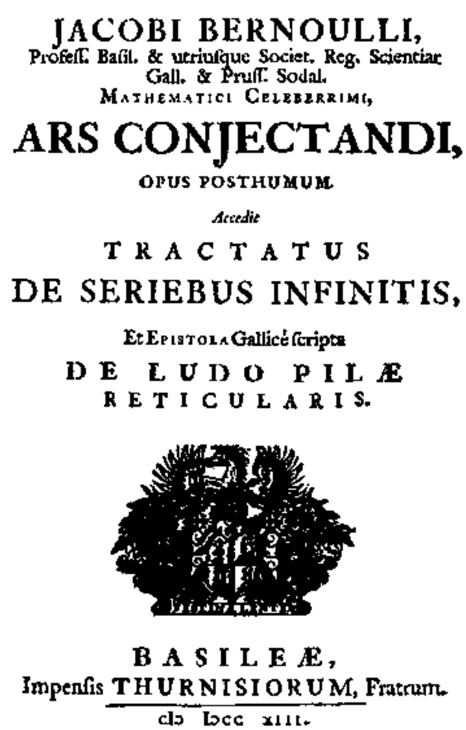


## Bernoulli numbers

September 2005

As we learned in last month's column, in the 1760's Euler wrote only two articles on series. There was E-326, a paper on the so-called central trinomial coefficients and the subject of that column. In researching that column, I also looked at the other paper, E-393, *De summis serierum numeros Bernoullianos involventium*, or "On sums of series involving Bernoulli numbers." While the paper itself didn't turn out to be all that interesting, the path that leads to the paper is fascinating. It is the story of the Bernoulli numbers.

Bernoulli numbers are a sequence of rational numbers that arise in a dazzling variety of applications in analysis, numerical analysis and number theory. When Charles Babbage designed the Analytical Engine in the 19th century, one of the most important tasks he hoped the Engine would perform was the calculation of Bernoulli numbers.



JACOBI BERNOULLI,
Profess. Basil. & utriusque Societ. Reg. Scientiar. Gall. & Pruss. Sodal.
MATHEMATICI CELEBERRIMI,
ARS CONJECTANDI,
OPUS POSTHUMUM.
Accedit
TRACTATUS
DE SERIEBUS INFINITIS,
Et EPISTOLA Gallicè scripta
DE LUDO PILÆ
RETICULARIS.

BASILEÆ,
Impensis THURNISIORUM, Fratrum.
cIↃ IↃcc xiii.

The first few Bernoulli numbers [K] are

$$B_0 = 1$$
$$B_1 = \frac{-1}{2}$$
$$B_2 = \frac{1}{6}$$
$$B_3 = 0$$
$$B_4 = \frac{-1}{30}$$
$$B_5 = 0$$
$$B_6 = \frac{1}{42}$$

After $B_1$ all Bernoulli numbers with odd index are zero, and the non-zero ones alternate in sign. They first appeared in 1713 in Jakob Bernoulli's pioneering work on probability, *Ars Conjectandi*. Jakob Bernoulli (1654-1705) was the older brother of Johann Bernoulli (1667-1748), who was, in turn, Euler's teacher and mentor at the University of Basel.

Sometimes people simply omit the Bernoulli numbers with odd index from the list, and write $B_k^*$ where we write $B_{2k}$. They, of course, must then make certain modifications to their formulas, and, in general, their formulas are a bit simpler.

Bernoulli was studying sums of powers of consecutive integers, like sums of squares,

$$1 + 4 + 9 + 16 + 25 = 55$$

or sums of cubes

$$1 + 8 + 27 + 64 + 125 + 216 + 343 = 784$$

In modern notation (Bernoulli did not use subscripts, nor did he use Σ for summations or ! for factorials) Bernoulli found that

$$\sum_{k=1}^{n-1} k^p = \sum_{k=0}^{p} \frac{B_k}{k!} \frac{p!}{(p+1-k)!} n^{p+1-k}$$

If $n$ is large and $p$ is small, that means that the left hand side is a sum of a relative large number of relatively small powers, and if we know the necessary Bernoulli numbers then the sum on the right is simpler to evaluate than the sum on the left. Bernoulli himself is said [G+S] to have used this formula to find the sum of the tenth powers of numbers 1 to 1000 in less than eight minutes. The answer is a 32-digit number.

Bernoulli numbers arise in Taylor series in the expansion

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!}.$$

Bernoulli numbers are also involved in the expansions of several other functions, including $\tan x$, $\frac{x}{\sin x}$, $\log\left(\frac{\sin x}{x}\right)$ and others.

Euler encountered Bernoulli numbers in his great solution to the Basel problem when he showed that

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \text{etc.} = \frac{p^2}{6},$$

though he did not recognize them at the time. In the same paper, Euler also evaluated

$$\sum_{k=1}^{\infty} \frac{1}{k^n}$$

for the first several even values of $n$. Only later would he realize that these other sums involved the Bernoulli numbers. In fact, if $n$ is even, then

$$\sum_{k=1}^{\infty} \frac{1}{k^n} = \frac{2^n |B_n| p^n}{2(n!)}.$$

Euler also failed to recognize the Bernoulli numbers in 1732 when he first did his work on the Euler-Maclaurin formula. Maclaurin also missed them when he discovered the formula independently in 1742. Again in modern form, the result says that for sufficiently smooth functions $f$, a series based on $f$ and an integral of $f$ are related by

$$\sum_{k=1}^{n} f(k) = \int_1^n f(x)\,dx + \frac{f(1)+f(n)}{2} + \sum_{k=1}^{p} \frac{B_{2k}}{(2k)!}\left( f^{2k-1}(n) - f^{2k-1}(1) \right) + R_n(f, p)$$

where $R_n(f, p)$ is a remainder term that usually disappears rapidly as $p$ increases. [G+S] The formula can be used either to estimate the series on the left knowing the integral on the right, or conversely, to estimate the integral by evaluating the series. Euler used the series on the left hand side of this formula in 1732 to estimate the values of infinite series and to find $\sum_{k=1}^{\infty} \frac{1}{k^2}$ to six decimal places. This is also how he found the first properties of $g$, the so-called Euler constant. Maclaurin used the other side of the formula to estimate the values of integrals from series.

The Bernoulli numbers are related to Euler's constant $\gamma$ by

$$g = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{B_{2k}}{2k}.$$

There is also an astonishing result due to Kummer [G+S] relating Bernoulli numbers to Fermat's Last Theorem. Kummer noticed that a prime number $p$ is special if it does not divide the numerators of any of the Bernoulli numbers $B_2$, $B_4$, $B_6$, … $B_{p-3}$. Such primes are now called *regular*. The first prime that is *not* regular is 37. Kummer showed that if $p$ is a regular prime, then Fermat's Last Theorem, that $x^p + y^p = z^p$ has no non-trivial integer solutions, is true for $p$.

In 1921, Eric Temple Bell, author of the well-known popular mathematics history book *Men of Mathematics*, proved [B]:

> Theorem: If $p$ is an odd prime which does not divide $4^r - 1$, then the numerator of $B_{2pr}$ is divisible by $p$.

There must be thousands of such results, and Bernoulli numbers continue to be studied today. JSTOR reports over 150 "hits" on the key words "Bernoulli numbers" since 1990.

Let us turn now to 1755, when Euler published his *Institutiones calculi differentialis* [E212]. At that time, only a few of the results above were known, and their links to Bernoulli numbers were apparently not yet recognized. The results that were known seem to be:

1. Bernoulli's own results on summing powers of integers. Bernoulli showed how this involved Bernoulli numbers, hence the name.
2. The Euler-Maclaurin summation formula.
3. Taylor series for various functions.
4. Euler's evaluation of $\zeta(2n)$.

Then, through all the trees, Euler sees the forest. It must have been a wonderful feeling to see how so many different aspects of mathematics are linked through these mysterious Bernoulli numbers.

Euler devotes almost all of chapters 5 and 6 of Part 2 of his *Calculus differentialis* to results related to Bernoulli numbers, and on page 420 (page 321 of the *Opera Omnia* edition) he attributes them to Jakob Bernoulli and calls them *Bernoulli numbers*. Unfortunately, only Part 1 of the *Calculus differentialis* has been translated into English, so readers who want to enjoy it in Euler's words must either brave the Latin or find a copy of the rare 1790 German translation.

Euler begins his chapter 5, "Investigation of the sums of series from their general term" with a quick treatment of Bernoulli's results on summing sequences of powers. Then he repeats his own results from the 1730's [E25] on the Euler-Maclaurin formula and gives the recursive relation on the coefficients in that formula. Euler doesn't mention Maclaurin, so he is probably unaware of his work on the subject.

Then he shows how those coefficients arise from the Taylor series expansions of $\frac{x}{1-e^{-x}}$ and $\frac{1}{2}\cot\left(\frac{1}{2}x\right)$.

Eventually, after quite a bit of work, he lists the Bernoulli numbers, naming them after Bernoulli in the process, and shows how they are related to the coefficients in the Euler-Maclaurin formula.

This done, he extends occurrence of Bernoulli numbers in the expansion of $\frac{1}{2}\cot\left(\frac{1}{2}x\right)$ to the more general form $\frac{p}{n}\cot\left(\frac{mp}{n}\right)$ and uses that to relate Bernoulli numbers to the values of $\zeta(2n)$. To end the theoretical parts of his exposition, he gives some of the properties of the Bernoulli polynomials and notes that Bernoulli numbers grow faster than any geometric series.

Euler spends the rest of these two chapters doing applications of Bernoulli numbers, including calculating the Euler-Mascheroni constant, $\gamma$, to 15 decimal places.

All this is rather unexpected in a textbook on differential calculus.

With this, Euler did not write again on Bernoulli numbers until 1768. In fact, in the intervening 13 years, he wrote only four papers on series. Besides the one on central trinomial coefficients that was the subject of last month's column, he wrote one paper on approximating pi, one on trig functions, and one on continued fractions.

We have already said that the 1768 paper, E-393, "didn't turn out to be all that interesting," but it might be worth summarizing its results. Euler opens E-393 with a list of Bernoulli numbers and a list of the coefficients that arise in $\zeta(2n)$, and shows how two lists are related. Then he gives his recursive relation on the zeta coefficients.

Then he leaps to the Euler-Maclaurin formula. Up to this point, most of the essay is just a new version of what he had presented in the *Calculus differentialis*. From here, though, he gives a different way to show the relation between the Bernoulli numbers and the expansion of $\frac{1}{2}-\frac{1}{2}\cot\left(\frac{1}{2}x\right)$. Then he uses this same technique to give new relations between the Bernoulli numbers and a variety of other functions and numbers, including $\frac{x}{2}\frac{e^{y}+e^{-y}}{e^{y}-e^{-y}}-\frac{1}{2}$ and $\frac{1}{e^{2}-1}$. Finally, he gives the values of integrals like $\int_0^1 \frac{(\ln x)^n}{(1-x)}dx$, for $n$ odd, in terms of the $(n + 1)^{st}$ Bernoulli number. None of this would have been appropriate to include in the *Calculus differentialis*.

Bernoulli numbers are still a bit mysterious. They appear frequently in Julian Havel's recent book *Gamma*, about Euler's constant, and people continue to discover new properties and to publish articles about them.

Simon Singh [S] quotes Andrew Wiles as describing the process of mathematical discovery with the colorful words "You enter the first room of the mansion and it's completely dark. You stumble around bumping into the furniture but gradually you learn where each piece of furniture is. Finally, after six months or so, you find the light switch, you turn it on, and suddenly it's all illuminated." It must have been something like this for Euler, when he saw how the "furniture" was arranged around the Bernoulli numbers.

In Euler's time, though, light switches hadn't yet been invented.

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*How Euler Did It* is updated each month.