How Euler Did It

by Ed Sandifer

Goldbach's series

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Christian Goldbach is a fairly well known, but rather minor figure in the history of mathematics. Pictures of Goldbach seem very rare, if indeed any survive. His name, if not his face, is widely known because it is attached to his conjecture that every even number larger than two is the sum of two prime numbers. Of the great unsolved problems in mathematics, the Goldbach Conjecture is probably the easiest to explain to a non-mathematician. Try explaining the Riemann hypothesis, for example, to a fourth grader or an English major.

Goldbach was also a kind of mentor to Leonhard Euler. For over 25 years they exchanged letters, 196 of which survive. These letters give us a window into Euler's scientific and personal life. In Goldbach's very first letter to Euler, dated December 1, 1729, Goldbach got Euler interested in number theory. Goldbach added note at the end of the letter: "P. S. Have you noticed the observation of Fermat that all numbers of the form $2^{2^{x-1}}+1$, that is 3, 5, 17, etc., are prime numbers, but he did not dare to claim he could demonstrate it, nor, as far as I know, has anyone else been able to prove it." This was mentioned in this column in November 2003.

Euler was not the only one to benefit from Goldbach's attentions. Earlier in 1729, Goldbach posed two series to Euler's friend Daniel Bernoulli. Those series were:

the sum of the reciprocals of the numbers one less than a power of two, that is:

$$1+\frac{1}{3}+\frac{1}{7}+\frac{1}{15}+\ldots+\frac{1}{2^n-1}+\text{etc.}, \text{ and}$$

the sum of the reciprocals of the numbers one less than powers, that is:

$$\frac{1}{3}+\frac{1}{7}+\frac{1}{8}+\frac{1}{15}+\frac{1}{24}+\frac{1}{26}+\text{etc.}$$

Euler mentions both of these series in E-25, written in 1732 and published in 1738. Sometime after Euler wrote that paper and before 1737 when he wrote E-72, Euler learned that Goldbach had found a way to sum the second of these series. In this column, we will look at some of the results in that paper, E-72, "Variae observationes circa series infinitas," or "Several observations about infinite series." We will see what Goldbach did, and see how much more Euler could do with the same idea. We will save some of the results for a later column.

E-72 is Euler's first paper that closely follows the modern Theorem-Proof format. There are no definitions in the paper, or it would probably follow the Definition-Theorem-Proof format. After an introductory paragraph in which Euler tells part of the story of the problem, Euler gives us a theorem and a proof:

"Theorem 1. This infinite series, continued to infinity,

$$\frac{1}{3}+\frac{1}{7}+\frac{1}{8}+\frac{1}{15}+\frac{1}{24}+\frac{1}{26}+\frac{1}{31}+\frac{1}{35}+\text{etc.}$$

the denominators of which are all numbers which are one less than powers of degree two or higher of whole numbers, that is, terms which can be expressed with the formula $\frac{1}{m^n-1}$, where m and n are integers greater than one, then the sum of this series is = 1."

Euler's proof begins with an 18th century step that treats infinity as a number. Such steps became unpopular among rigorous mathematicians about a hundred years later. He takes x to be the sum of the harmonic series

$$x=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\text{etc.}$$

Today, Euler would have learned in calculus that we are not allowed to treat x as an infinite number like this, and he would not have discovered this remarkable proof.

Next, Euler subtracts the geometric series

$$1=\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\text{etc.}$$

leaving

$$x-1=1+\frac{1}{3}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{9}+\frac{1}{10}+\text{etc.}$$

Subtract another geometric series

$$\frac{1}{2}=\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\frac{1}{243}+\text{etc.}$$

leaving

$$x-1-\frac{1}{2}=1+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{10}+\frac{1}{11}+\text{etc.}$$

and another geometric series

$$\frac{1}{4}=\frac{1}{5}+\frac{1}{25}+\frac{1}{125}+\text{etc.}$$

leaving

$$x-1-\frac{1}{2}-\frac{1}{4}=1+\frac{1}{6}+\frac{1}{7}+\frac{1}{10}+\text{etc.''}$$

Note that Euler had to skip subtracting the geometric series

$$\frac{1}{3}=\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\frac{1}{256}+\text{etc.}$$

because the series of powers of $\frac{1}{4}$ on the right is already a subseries of the series of powers of $\frac{1}{2}$, so those terms have already been subtracted. This happens because 3 is one less than a power, 4. It happens again every time we reach a term one less than a power. He will have to skip 7, because that is one less than the cube 8, 8 because it is one less than the square 9, 15 because it is one less than the square 16, etc.

Continuing in this way, we see that all of the terms on the right except the term 1 can be eliminated, leaving

$$x-1-\frac{1}{2}-\frac{1}{4}-\frac{1}{5}-\frac{1}{6}-\frac{1}{9}-\text{etc.}=1$$

so

$$x-1=1+\frac{1}{2}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{9}+\frac{1}{10}+\text{etc.}$$

Now it gets just a little bit tricky. Since x is sum of the harmonic series, Euler believes that the 1 on the left must equal the terms of the harmonic series that are missing on the right. Those missing terms are exactly the ones with denominators one less than powers, so Euler concludes that

$$1=\frac{1}{3}+\frac{1}{7}+\frac{1}{8}+\frac{1}{15}+\frac{1}{24}+\frac{1}{26}+\text{etc.}$$

where the terms on the right have denominators one less than powers.

Q. E. D.

This is a beautiful proof, and a true result. Unfortunately, the beautiful proof is of the form "infinity minus infinity equals 1", and so is not valid. Bruce Burdick, of Roger Williams University, has discovered, but not yet published, a direct proof, using the principle of inclusion and exclusion, that meets modern standards of rigor, but it somehow lacks the spirit of Goldbach's proof, as presented by Euler. Also, Pelegrini Viader of Barcelona has discovered a proof that uses nonstandard analysis. He says that his paper will appear soon.

This seems to be as far as Goldbach took this problem, and it's a little hard to know how much of this proof is Euler's and how much is Goldbach's. Euler has more ideas. He gives us

Theorem 2: $\frac{1}{3}+\frac{1}{7}+\frac{1}{15}+\frac{1}{31}+\frac{1}{35}+\frac{1}{63}+\text{etc.}=l\,2$.

Here the denominators are one less than the powers of even numbers, and Euler, as we have seen before, writes $l2$ to denote what we would write ln 2.

In Theorem 3, he tells us that

$$\frac{p}{4}=1-\frac{1}{8}-\frac{1}{24}+\frac{1}{28}-\frac{1}{48}-\frac{1}{80}-\frac{1}{120}-\frac{1}{124}-\frac{1}{168}-\frac{1}{224}+\frac{1}{244}-\frac{1}{288}-\text{etc.}$$

where the denominators are "evenly even" numbers, that is numbers that are divisible by four, that are one more or one less than the powers of odd numbers, and where the denominators that are one more than powers have a + sign, while the ones one less have a – sign.

Theorems 4, 5 and 6 are even more exotic series.

We are only about half way through this paper, and there are remarkable things to come. Euler will adapt these same methods to develop theorems relating infinite sums to infinite products, prove an important result about what we now call the Riemann zeta function, and give an entirely new proof of the result, first given by Euclid, that there are infinitely many prime numbers. We will examine these results in a later column.

References:

[D] Dunham, William, *Euler The Master of Us All*, Dolciani Mathematical Expositions vol. 22, Mathematical Association of America, Washington, DC, 1999.

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How Euler Did It is updated each month.