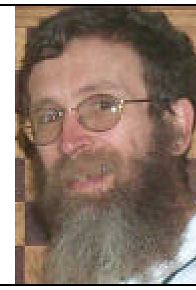
MAA Online

How Euler Did It

by Ed Sandifer



A Mystery about the Law of Cosines

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The Law of Cosines has been in newspapers and magazines lately. Perhaps you have seen an advertisement that reads "Margaret needs to know what the heck $a^2 - 2ab\cos q + b^2$ is all about." They are trying to recruit people to teach high school mathematics. Those of us who recognize Margaret's formula as part of the Law of Cosines would make good candidates.

The day I first saw this advertisement, I had been reading parts of Leonhard Euler's *Calculus Integralis*, published in 1768. There, I found this same form in a very different context, and I thought it was mysterious.

Euler wrote a three-volume text on integral calculus. The volumes appeared in 1768, 1769 and 1770 and bear Eneström numbers 342, 366 and 385. Together with the two volumes of the *Introductio in analyisin infinitorum*, E 100 and 101, published in 1748, and the *Calculus differentialis*, E 212, 1755, and weighing in at over 2500 pages, they form the first really thorough set of calculus textbooks. They are often described as forming the basis for the modern calculus curriculum. This is something of an exaggeration, though. Much of the modern curriculum is missing from Euler. For example, Euler does no applications outside mathematics. There are no related rates problems or problems in physics. In fact, there are no exercises at all. On the other hand, Euler includes much that is not in most modern calculus courses. He does a lot of differential equations that we usually do in a separate course. Volume III ends with a long chapter on the Calculus of Variations, and Chapter 6 of the second part of Volume I covers a good deal about Elliptic Integrals, including the segment addition theorem. Both of these topics are very rare in the modern calculus curriculum.

Each volume has two or three "parts", and each "part" has about ten "chapters." A typical chapter consists of several "problems," each followed by a solution and several corollaries and scholions. Chapter 2 of the first part of Volume I is titled "De integratione formularum differentialium irrationalium," which translates as "On the integration of irrational differential formulas." It opens with:

Problem 6. Given a differential formula $dy = \int \frac{dx}{\sqrt{a + bx + gxx}}$, to find its integral.

Euler's solution considers two cases. The first case is that the quadratic has two distinct real roots, and the second is that the quadratic is irreducible. Euler does not consider what might be a third case, that the quadratic has two equal roots, for in that case, taking the square root in the denominator reduces he problem to a much easier problem.

Euler's first case is not that important to the point we want to make in this column, so we will sketch it very briefly. He supposes that the quadratic factors into $(a+bx)(c+dx)$. Then he takes $z=\sqrt{\dfrac{f+gx}{a+bx}}$ and rewrites the quadratic as $(a+bx)^2\,zz$. He shows how the integral involves logarithms if the signs of a and g are the same, and involves trigonometric functions if they have different signs.

What interests us is when he says that if the quadratic doesn't factor, then he can write it as

$$dy=\frac{dx}{\sqrt{aa-2abx\cos z+bbxx}}$$

If we take $x = 1$, and substitute one Greek letter for another, then the expression inside the radical in the denominator is part of that Law of Cosines that the advertisement says Margaret needs to know! Who would have thought that irreducible quadratics would have anything to do with the Law of Cosines?

Is this quadratic really irreducible? Its discriminant is $a^2b^2(\cos^2 z-1)$, and that is never positive. When $\cos z=1$, the discriminant is zero and there is a double root, and we have already excluded that case. So, indeed, it is safe to say the quadratic really is irreducible.

Other questions remain. Can any irreducible quadratic be put in this form? What is the significance of the angle ζ? What are the roots of such a form? Why would Euler use such a form?

Readers who have pencils are encouraged to investigate these questions a bit before reading any farther.

There are a number of ways to approach this challenge, each with its own beautiful aspects. I posed this at dinner at a recent MAA Section meeting, and no two people at the table of six did it the same way. My favorite involved completing the square and an application of Euler's formula, $e^{iq}=\cos q+i\sin q$.

Rather than deprive the reader of the pleasure of discovering such pleasant derivations, I'll describe a less elegant approach. Suppose we are given the roots of an irreducible quadratic in polar form, say $(r,\pm q)$. Then we can write the roots in Cartesian form, as $s=r\cos q+i\sin q$ and $t=r\cos q-i\sin q$. When we expand $(x-s)(x-t)$, we get $x^2-2r\cos q+r^2$. From this, it is easy to see that any irreducible quadratic polynomial can indeed be put into Euler's form. In fact, just as it is easy to see that the roots are s and t when we write it in the form $a(x-s)(x-t)$, it is easy to see that, in polar form (and when a and b have the same sign), the roots of $aa-2abx\cos z+bbxx$ are $\left(\dfrac{a}{b},\pm z\right)$. (The case of mixed signs is only slightly irksome.)

It might seem that we've solved most of the mystery; why is the form $aa-2abx\cos z+bbxx$ irreducible? But in finding this answer, we have made a serious error in historical analysis. We have represented complex numbers in polar form, using a radius and an angle. This idea is usually said to have originated in a 1797 paper by Caspar Wessel [N, p. 48], so Euler, in 1768, should not have been able to use it.

So, we are left with an even more perplexing mystery. Euler uses an idea that we understand because we know about the polar form of a complex number. Euler had no such notion, yet he uses the idea as if it were natural and well known in its time. How did Euler know and understand that all irreducible quadratics could be written as $aa - 2abx\cos z + bbxx$?

I don't know. And I also don't know how it is connected to the Law of Cosines. Maybe our young friend Margaret will be the one to figure it out.

References:

[E342] Euler, Leonhard, *Institutionum calculi integralis volumen primum*, St. Petersburg, 1768, reprinted in *Opera Omnia* Series I vol 11.

[N] Nahin, Paul, *An Imaginary Tale: The Story of* $\sqrt{-1}$, Princeton Univ. Press, 1998.

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How Euler Did It is updated each month.