Invariants from noncommutative index theory for homotopy equivalences

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Basics in noncommutative index theory: A noncommutative Chern character

Given

- $\mathcal{A}$ C^*-algebra p. e. $\mathcal{A} = C(B)$, B closed manifold
- $\mathcal{A}_\infty \subset \mathcal{A}$ "smooth" subalgebra (=closed under holomorphic functional calculus, dense, etc.) $\mathcal{A}_\infty = C^\infty(B)$

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one gets

- a $\mathbb{Z}$-graded Fréchet algebra $\hat{\Omega}_*\mathcal{A}_\infty$ of noncommutative differential forms with differential $\mathrm{d} : \hat{\Omega}_k\mathcal{A}_\infty \to \hat{\Omega}_{k+1}\mathcal{A}_\infty$
- a Chern character $\mathrm{ch} : K_*(\mathcal{A}) \to H_*^{dR}(\mathcal{A}_\infty)$
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Motivating example used in higher index theory: Γ finitely generated group with length function, $\mathcal{A} = C^*\Gamma$, $\mathcal{A}_\infty$ the Connes-Moscovici algebra

Dirac operators over C^*-algebras

Given

- (M, g) closed oriented Riemannian manifold
- $E \to M$ hermitian bundle with Clifford action and compatible connection ($\mathbb{Z}/2$-graded, if dim M even)
- $P \in C^\infty(M, M_n(\mathcal{A}_\infty))$ projection

we get an $\mathcal{A}$-vector bundle $\mathcal{F} := P(\mathcal{A}^n \times M) \to M$
and a (odd) Dirac operator $D_\mathcal{F} : C^\infty(M, E \otimes \mathcal{F}) \to C^\infty(M, E \otimes \mathcal{F})$.

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Important example for higher index theory:

the Mishenko-Fomenko vector bundle: $\mathcal{F} = \tilde{M} \times_\Gamma C^*\Gamma$ with $\Gamma = \pi_1(M)$.

$E = S$ the spinor bundle (gives twisted spin Dirac operator)

$E = \Lambda^*(T^*M)$ (gives twisted de Rham or signature operator).

Index theory

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Proposition (Atiyah-Singer index theorem)

$$\mathrm{ch}(\mathrm{ind}(D_{\mathcal{F}})) = \int_M \hat{A}(M)\, \mathrm{ch}(E/S)\, \mathrm{ch}(\mathcal{F}) \quad \in H_*^{dR}(\mathcal{A}_\infty)\ .$$

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Application in higher index theory: If $D_{\mathcal{F}}$ is the signature operator twisted by $\mathcal{F} = \tilde{M} \times_\Gamma C^*\Gamma$, then $\mathrm{ind}(D_{\mathcal{F}})$ is homotopy invariant.

The proposition implies: By pairing $\mathrm{ch}(\mathrm{ind}(D_{\mathcal{F}}))$ with cyclic cocycles one gets higher signatures.

This can be used to prove the Novikov conjecture for Gromov hyperbolic groups (Connes-Moscovici 1990).

Secondary invariants

Let A be a smoothing symmetric operator on $L^2(M, E \otimes \mathcal{F})$ such that $D_{\mathcal{F}} + A$ is invertible. (A should be odd if $\dim M$ is even.)

Then one can define

$$\eta(D_{\mathcal{F}}, A) \in \hat{\Omega}_* \mathcal{A}_\infty / \overline{[\hat{\Omega}_* \mathcal{A}_\infty, \hat{\Omega}_* \mathcal{A}_\infty] + \mathrm{d}\, \hat{\Omega}_* \mathcal{A}_\infty}$$

generalizing the classical η-invariant (with $\mathcal{A} = \mathbb{C}$)

$$\eta(D_{\mathcal{F}}, A) = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-\frac{1}{2}} \mathrm{Tr}(D_{\mathcal{F}} e^{-t(D_{\mathcal{F}}+A)^2}) dt \ .$$

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Higher η-invariants were introduced by Lott (1992). The general definition is implicit in work of Lott (1999).

Atiyah-Patodi-Singer index theorem

Let M be an oriented Riemannian manifold with cylindric end $Z = \mathbb{R}^+ \times \partial M$. On Z all structures are assumed of product type.

If $\dim M$ is even, then on Z

$$D^+_{\mathcal{F}} = c(dx)(\partial_x - D^{\partial}_{\mathcal{F}}) \ .$$

Let A be a symmetric, smoothing operator on $L^2(\partial M, E^+ \otimes \mathcal{F})$ such that $D^{\partial}_{\mathcal{F}} + A$ is invertible.

Let $\chi : M \to \mathbb{R}$ be smooth, $\operatorname{supp} \chi \subset Z$; $\operatorname{supp}(\chi - 1)$ compact.

Proposition (W., 2009)

$$\operatorname{ch} \operatorname{ind}(D^+_P - c(dx)\chi(x)A) = \int_M \hat{A}(M) \operatorname{ch}(E/S) \operatorname{ch}(\mathcal{F}) - \eta(D^{\partial}_{\mathcal{F}}, A) \ .$$

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The proposition generalizes the higher APS index theorem proven by Leichtnam-Piazza (1997-2000).

Higher ρ-invariants for homotopy equivalences

Let M, N be odd-dimensional oriented closed Riemannian manifolds, $f : M \to N$ a smooth orientation preserving homotopy equivalence.

$\mathcal{A} = C^*\Gamma$, $\Gamma = \pi_1(N)$.

$\mathcal{F}_N = \tilde{N} \times_\Gamma C^*\Gamma$ Mishenko-Fomenko bundle, $\mathcal{F}_M = f^*\mathcal{F}_N$.

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Let $\hat{\Omega}_*^{<e>}(\mathcal{A}_\infty) = \mathbb{C} < g_0 \, \mathrm{d}\, g_1 \dots \mathrm{d}\, g_m \mid g_0 g_1 \dots g_m = e > \subset \hat{\Omega}_* \mathcal{A}_\infty$.

Definition

$$\rho(f) := [\eta(D_\mathcal{F}, A)] \in \hat{\Omega}_*\mathcal{A}_\infty / \overline{[\hat{\Omega}_*\mathcal{A}_\infty, \hat{\Omega}_*\mathcal{A}_\infty] + \mathrm{d}\, \hat{\Omega}_*\mathcal{A}_\infty + \hat{\Omega}_*^{<e>}\mathcal{A}_\infty} \, .$$

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6. (Product formula) If $N = N_1 \times X$, $M = M_1 \times X$, $f = f_1 \times \mathrm{id}_X$, then $\rho(f) = \rho(f_1)\,\mathrm{ch}(\mathrm{ind}(D_{\mathcal{F}_X}))$.

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The proofs use the APS index theorem. Local terms vanish since we divided out $\hat{\Omega}_*^{<e>}\mathcal{A}_\infty$.

(4) is based on a generalization of results of Hilsum-Skandalis (1992) to manifolds with cylindric ends.

(6) uses a product formula for noncommutative η-forms (W., 2009).

Applications

dim $N = 4k - 1$, $k \geq 2$, $\Gamma = \pi_1(N)$ not torsion free

Proposition (Chang-Weinberger 2003)

There are homotopy equivalences $f_i : M_i \to N, i \in \mathbb{N}$ such that
$\rho_{L^2}(M_i) \neq \rho_{L^2}(M_j)$, *$i \neq j$.*
Thus $[(M_i, f_i)]$ are distinct in $\mathcal{S}(N)$.

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Corollary

Let X *be a closed manifold with a non-zero higher signature. Assume that* $\pi_1(X), \Gamma$ *are Gromov hyperbolic.*
Then $[(M_i \times X, f_i \times \mathrm{id}_X)]$ *are distinct in* $\mathcal{S}(N \times X)$ *and distinguished by* $\rho(f_i \times \mathrm{id}_X)$.

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Idea of proof: A non-zero higher signature implies that $\mathrm{ch}\,\mathrm{ind}(D_{\mathcal{F}_X}) \neq 0$.
Now apply product formula to $\rho(f_i \times \mathrm{id})$.

Open questions I

$\Gamma = \pi_1(N)$

Does the following diagram commute?

$$\begin{array}{ccc} L_{n+1}(\mathbb{Z}\Gamma) & \longrightarrow & \mathcal{S}(N) \\ \downarrow & & \downarrow \rho \\ K_{n+1}(C^*\Gamma) & \xrightarrow{\text{ch}} & \hat{\Omega}_*\mathcal{A}_\infty / \overline{[\hat{\Omega}_*\mathcal{A}_\infty, \hat{\Omega}_*\mathcal{A}_\infty] + \mathrm{d}\,\hat{\Omega}_*\mathcal{A}_\infty + \hat{\Omega}_*^{<e>}\mathcal{A}_\infty} \end{array} .$$

A positive answer to this question would lead to more general applications.

Open questions II

What is the connection with

- the Higson-Roe map "from surgery to analysis" (2004)?

$$\begin{array}{ccccccc}
L_{n+1}(\mathbb{Z}\Gamma) & \longrightarrow & \mathcal{S}(N) & \longrightarrow & \mathcal{N}(N) & \longrightarrow & L_n(\mathbb{Z}\Gamma) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
K_{n+1}(C_r^*\Gamma) & \longrightarrow & K_{n+1}(D_\Gamma^* N) & \longrightarrow & K_n(B\Gamma) & \longrightarrow & K_n(C_r^*\Gamma)
\end{array}$$

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- the Higson-Roe interpretation of the APS ρ-invariant (2010)?

$$\begin{array}{ccccc} K_{n+1}(C^*\Gamma) & \longrightarrow & \varinjlim_{X \subset B\Gamma} K_{n+1}(D^*_\Gamma X) & \longrightarrow & K_n(B\Gamma) \\ \downarrow & & \downarrow \rho_{APS} & & \downarrow \\ \mathbb{Z} & \longrightarrow & \mathbb{R} & \longrightarrow & \mathbb{R}/\mathbb{Z} \end{array}$$

Some literature

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