The Higson-Mackey Analogy for Complex Semisimple Groups and their Finite Extensions

John R. Skukalek
American University

ECOAS 2010

Almost Connected Groups and the Baum-Connes Conjecture

- G: locally compact, second-countable group
- G_e: connected component of the identity $e \in G$.
- Suppose that G is almost connected, i.e. G/G_e is compact.
- K: maximal compact subgroup of G.
- G/K is a universal example for proper actions of G.

- Theorem (Chabert-Echterhoff-Nest, 03)

  The Baum-Connes assembly map

$$\mu : K_*^G(G/K) \to K_*(C_\lambda^*(G))$$

  is an isomorphism.

Connected Lie Groups and the Baum-Connes Conjecture

- G: connected Lie group.
- μ can be defined using "Dirac induction" from K (Kasparov).
- In this setting, Baum-Connes conjecture (without coefficients) known as Connes-Kasparov conjecture.
- Progress in the early-mid 80's:
  - Simply connected solvable groups (Connes).
  - Nilpotent groups (Rosenberg).
  - Amenable groups (Kasparov).
  - Complex semisimple groups (Pennington and Plymen).
  - Linear reductive groups (Wassermann).
- Latter two cases make use of detailed representation theory.

Continuous Fields and the Baum-Connes Conjecture

- G: almost connected Lie group (G/G_e finite)
- $G/K \cong \mathfrak{g}/\mathfrak{k}$, the quotient of the Lie algebras of G and K.
- Domain of assembly map identifies with $K_*(K \ltimes \mathfrak{g}/\mathfrak{k})$ (Green-Julg, Kasparov).
- Smooth one-paramter family of Lie groups (Lie groupoid)

$$G_t = \begin{cases} K \ltimes \mathfrak{g}/\mathfrak{k} & \text{if } t = 0 \\ G & \text{if } t \neq 0. \end{cases}$$

- Continuous field of C^*-algebras $\{C^*_\lambda(G_t)\}_{t\in[0,1]}$ produces an "asymptotic" assembly map

$$\mu_0 : K_*(C^*(G_0)) \to K_*(C^*_\lambda(G)).$$

- Noted in Baum-Connes-Higson paper and proved in Connes' book (94):

$$\mu \text{ is an isomorphism } \iff \mu_0 \text{ is an isomorphism.}$$

The Mackey Analogy

- G: connected semisimple Lie group (finite center).
- $G_0 = K \ltimes \mathfrak{g}/\mathfrak{k}$ known as a Cartan motion group.
- E.g. $G = SL(n, \mathbb{F})$, $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$
  - $G_0 = K \ltimes \mathfrak{p}$, where

$$\mathfrak{p} = \{A \in M_n(\mathbb{F}) | A^* = A, tr(A) = 0\}$$

and K is $SO(n)$ or $SU(n)$ acting by matrix conjugation.

Proposal (Mackey, 75)

There ought to be a "natural" correspondence between almost all irreducible tempered representations of G and almost all irreducible unitary representations of G_0:

$$\widehat{G}_\lambda \longleftrightarrow \widehat{G_0} \quad \text{a.e. Plancherel.}$$

The Mackey Analogy

- According to the "Mackey machine," the unitary dual of any semidirect product $K \ltimes A$ with K compact and A locally compact abelian is parametrized by:
  - characters $\chi : A \to \mathbb{T}$ of A.
  - irreducible representations of the isotropy subgroups $K_\chi \subseteq K$.
- G: connected complex semisimple Lie group (e.g. $SL(n, \mathbb{C})$).
- The tempered (reduced unitary) dual of G is parametrized by:
  - characters of the Borel subgroup B (e.g. upper Δ matrices).
- Higson observed that for $G_0 = K \ltimes \mathfrak{p}$, each K_χ is connected. Consequently, there is a canonical bijection

$$\Phi : \widehat{G}_\lambda \xrightarrow{\cong} \widehat{G_0}.$$

- Φ is not a homeomorphism.

Higson's Analysis

- Each $\pi \in \widehat{G}_\lambda \cup \widehat{G_0}$ contains a unique $\tau \in \widehat{K}$ satisfying a minimality condition a la Vogan.
- Φ preserves these minimal K-types.
- Each τ gives rise to a subquotient of $C^*_\lambda(G)$ and of $C^*(G_0)$.

- Theorem (Higson, 06)

  The above subquotients are Morita equivalent; in fact, to the same commutative C^*-algebra.

  - Thus $\widehat{G}_\lambda$ and $\widehat{G_0}$ can be partitioned into homeomorphic (locally closed) subsets.
  - An elaboration of this analysis to the continuous field $\{C^*_\lambda(G_t)\}_{t\in[0,1]}$ establishes, using nothing more K-theoretic than Bott periodicity, that

$$\mu_0 : K_*(C^*(G_0)) \to K_*(C^*_\lambda(G))$$

is an isomorphism.

The Almost Connected Case

- Consider an extension of groups

$$1 \to G \to \mathcal{G} \to F \to 1$$

in which G is connected complex semisimple and F is finite.

- So $\mathcal{G}$ has $|F|$ connected components and identity component G, but needn't be a complex Lie group.
- There are maximal compact subgroups satisfying

$$1 \to K \to \mathcal{K} \to F \to 1.$$

- Thus with $G_0 = K \ltimes \mathfrak{g}/\mathfrak{k}$ and $\mathcal{G}_0 = \mathcal{K} \ltimes \mathfrak{g}/\mathfrak{k}$ we have

$$1 \to G_0 \to \mathcal{G}_0 \to F \to 1.$$

- F acts on $\widehat{G}_\lambda$, $\widehat{G_0}$ and $\widehat{K}$.

The Almost Connected Case

- We have a commutative diagram

$$\begin{array}{ccc} \widehat{G}_\lambda & \xrightarrow[\cong]{\Phi} & \widehat{G_0} \\ & \searrow^{\tau} \quad \swarrow_{\tau_0} & \\ & \widehat{K} & \end{array}$$

  in which τ and τ_0 are maps assigning minimal K-types.

- Proposition

  The above diagram is F-equivariant.

  - A general version of the Mackey machine yields a bijection

$$\tilde{\Phi} : \widehat{\mathcal{G}}_\lambda \xrightarrow{\cong} \widehat{\mathcal{G}_0}$$

  such that $\pi \in \widehat{G}_\lambda$ occurs in $\tilde{\pi}|_G \iff \Phi(\pi)$ occurs in $\tilde{\Phi}(\tilde{\pi})|_{G_0}$.

Twisted Crossed Products

- G: locally compact group, C: C^*-algebra.
- $\alpha : G \to \mathrm{Aut}(C)$ continuous action
- (Green, 78) A *twisting map* for α is a (strongly) continuous homomorphism $\sigma : N \to \mathcal{UM}(C)$, where N is a closed normal subgroup of G, satisfying:
  1. $\alpha_n(c) = \sigma(n)c\sigma(n)^* \quad \forall\ n \in N,\ c \in C.$
  2. $\sigma(gng^{-1}) = \alpha_g[\sigma(n)] \quad \forall\ g \in G,\ n \in N.$
- Call (α, σ) a *twisted* action of G/N:
  - Ordinary action of G/N lifts to a twisted action of G/N whose twisting map is trivial.
- A covariant representation (U, π) of (G, C) *preserves* σ if

$$\pi(\sigma(n)) = U_n \quad \forall\ n \in N.$$

Twisted Crossed Products

- The *twisted* crossed product C^*-algebra

$$(G, N) \ltimes_{\alpha,\sigma} C.$$

  is the quotient of the ordinary crossed product $G \ltimes_\alpha C$ by the ideal corresponding to σ-preserving representations.
  - Completion of functions $G \to A$ such that

$$f(ng) = f(g)\sigma(n)^* \quad \forall\, g \in G, n \in N,$$

    with operations defined using G/N in place of G.
- If twisting map is trivial, then $(G, N) \ltimes C \cong G/N \ltimes C$.
- If $G'/N' \xrightarrow{\cong} G/N$, then restriction yields

$$(G, N) \ltimes C \xrightarrow{\cong} (G', N') \ltimes C.$$

- In particular, $(G, N) \ltimes C \cong G/N \ltimes C$ when $G \cong G/N \ltimes N$.

Twisted Crossed Products: Fundamental Example

- Assume for simplicity that G and N are unimodular.
- Twisted action of G/N on $C^*(N)$ given by

$$[\alpha_g(f)](n) = f(g^{-1}ng)$$

$$[\sigma(n')f](n) = f(n'^{-1}n)$$

  for all $g \in G$, $f \in C_c(N)$, and $n, n' \in N$.
- Associating to each $f \in C_c(G)$ $\tilde{f} : G \to C_c(N)$ defined by

$$[\tilde{f}(g)](n) = f(ng) \quad \forall\, g \in G, n \in N.$$

  yields an isomorphism

$$C^*(G) \xrightarrow{\cong} (G, N) \ltimes C^*(N).$$

Back to the Almost Connected Case

- $\mathcal{G}$: finite extension of connected complex semisimple group G.
- $K \subseteq \mathcal{K}$ maximal compact subgroups.
- $C^*_\lambda(\mathcal{G}) \cong (\mathcal{K}, K) \ltimes C^*_\lambda(G)$, $C^*(\mathcal{G}_0) \cong (\mathcal{K}, K) \ltimes C^*(G_0)$.
- $C_0(X_\tau)$, $\tau \in \widehat{K}$: Higson's commutative C^*-algebras from the connected case.
- To each $\mathcal{K}/K$-orbit $\mathcal{O} \subseteq \widehat{K}$ is associated a subquotient of $C^*_\lambda(\mathcal{G})$ and of $C^*(\mathcal{G}_0)$.

- Theorem

  The above subquotients are Morita equivalent to a twisted crossed product

$$(\mathcal{K}, K) \ltimes \bigoplus_{\tau \in \mathcal{O}} C_0(X_\tau, End(V_\tau)).$$

- Corollary

  $\mu_0 : K_*(C^*(\mathcal{G}_0)) \to K_*(C^*_\lambda(\mathcal{G}))$ *is an isomorphism.*

Other Classes of Lie groups?

- Theorem (George, 09)
  There exists a bijection
  $$\widehat{G}_\lambda \cong \widehat{G_0}$$
  for $G = SL(n, \mathbb{R})$ *that preserves minimal* K*-types.*
  - For more on real reductive groups, ask Nigel.
  - Question: Can one prove Baum-Connes for simply connected nilpotent groups using Kirilov's orbit method?
  - If G is (finite extension of) connected complex semisimple, $SL(n, \mathbb{R})$, or simply connected nilpotent, we have a bijection
  $$\widehat{G}_\lambda \cong \widehat{G_0}/N_G(K).$$
  - In the semisimple case, $N_G(K) = K$ acts trivially on $\widehat{G_0}$.
  - In the nilpotent case, $K = \{e\}$ so $N_G(K) = G$ and $\widehat{G_0} \cong \mathfrak{g}^*$.

Appendix A: Higson's Bijection

- $\widehat{G}_\lambda \cong \widehat{H}/W$ where $H \subseteq B$ is a Cartan subgroup of G.
- $H = MA$ where $A = \exp(\mathfrak{a})$, $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{p}$, $M = Z_K(A)$ is a maximal torus in K, and $W = N_K(A)/M$ is the Weyl group of G.
- $\mathfrak{p} = \mathfrak{a} \oplus \mathfrak{a}^\perp$: M-invariant decomposition.
- $\forall\ \varphi \in \widehat{A} \cong \mathfrak{a}^*$, $\exists\ w \in W$ such that $w \cdot \varphi$ is trivial on $\mathfrak{a}^\perp$.
- $K_{w\cdot\varphi} \subseteq K$ is connected with maximal torus M and Weyl group $W_\varphi \subseteq W$.
- The map

$$\mathrm{Ind}_B^G\ \sigma \otimes \varphi \mapsto \mathrm{Ind}_{K_{w\cdot\varphi} \ltimes \mathfrak{p}}^{G_0}\ \tau_\sigma \otimes (w \cdot \varphi)$$

where $\tau_\sigma \in \widehat{K_{w\cdot\varphi}}$ has highest weight σ establishes a bijection

$$\widehat{G}_\lambda \xrightarrow{\cong} \widehat{G_0}.$$

Appendix B: The Twisted Action

- For each $[\tau] \in \widehat{K}$, choose a representative $\tau : K \to \mathcal{U}(V_\tau)$.
- Suppose $y \in \mathcal{K}$ is such that $y \cdot [\tau] = [\tau']$.
- There exists a unitary operator $U_y : V_\tau \to V_{\tau'}$ unique up to a factor in $\mathbb{T}$ satisfying

$$U_y \tau(k) U_y^* = \tau'(yky^{-1}) \quad \forall\, k \in K.$$

- Given $T \in \mathrm{End}(V_\tau)$, let

$$y \cdot T = U_y T U_y^* \in \mathrm{End}(V_{\tau'}).$$

- Given $f : X_\tau \to \mathrm{End}(V_\tau)$, define

$$\alpha_y(f) : X_{\tau'} \to \mathrm{End}(V_{\tau'})$$

$$[\alpha_y(f)](\pi) = y \cdot f(y^{-1} \cdot \pi) \quad \forall\, \pi \in X_{\tau'}.$$

Appendix B: The Twisted Action

- Thus $\mathcal{K}$ acts via α on the C^*-algebra direct sum

$$\bigoplus_{\tau \in \mathcal{O}} C_0(X_\tau, \mathrm{End}(V_\tau)).$$

- K acts on each summand by

$$[\alpha_k(f)](\pi) = \tau(k) f(\pi) \tau(k)^*.$$

- Hence $\sigma(k)f = \tau(k) \circ f$ defines a twisting map, so that we obtain a twisted action (α, σ) of $\mathcal{K}/K$.