Amalgamated free products of C^*-algebras with MF property

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1 Residually Finite Groups

Definition 1.1

$$\begin{gathered} \textit{A countable group } G \textit{ is a residually finite (RF) group} \\ \Leftrightarrow \forall\ e \neq g \in G,\ \exists a \textit{ finite group } H \textit{ and a homomorphism } \rho : G \to H, \\ \textit{s.t. } \rho(g) \neq e. \\ \Leftrightarrow G \textit{ embeds into } \prod_k H_k \textit{ for a family of finite groups } \{H_k\}_k \end{gathered}$$

RF groups include: finite group, finite generated abelian group, free groups F_n, and $SL_n(\mathbb{Z})$.

Remark 1.2 *If G is a residually finite group, then*

$$L(G) \hookrightarrow \mathcal{R}^\omega$$

i.e. Connes' embedding problem for $L(G)$ has a "yes" answer.

Remark 1.3 *Connes embedding problem asks whether every separable II_1 factor can be embedded into $\mathcal{R}^\omega$, where $\mathcal{R}^\omega$ is the untrapower of the hyperfinite II_1 factor $\mathcal{R}$.*

Theorem 1.4 (Gruenberg, 1957) *Suppose G_1, G_2 are RF. Then $G_1 * G_2$ is RF.*

Theorem 1.5 (Baumslag, 1963) *Suppose that $G_1 \supseteq H \subseteq G_2$, where G_1, G_2 are RF and H is finite. Then the generalized free product of G_1 and G_2 with amalgamation over H, $G_1 *_H G_2$ is RF.*

An example of G. Higman in 1951 showed that $G_1 *_H G_2$ might not be RF when G_1, G_2 are RF and H is an infinite cyclic group. For example,

$$G_1 = \langle a, c : a^{-1}ca = c^2 \rangle; \quad G_2 = \langle b, c : b^{-1}cb = c^2 \rangle; \quad H = \langle c \rangle.$$

And

$$G_1 *_H G_2 = \langle a, b, c : a^{-1}ca = b^{-1}cb = c^2 \rangle$$

2 Residually Finite Dimensional C*-algebras

Definition 2.1

$$\begin{aligned} &\textit{A separable } C^*\textit{-algebras } \mathcal{A} \textit{ is residually finite dimensional (RFD)} \\ \Longleftrightarrow\ &\forall\ 0 \neq x \in \mathcal{A},\ \exists a \textit{ finite dimensional } C^*\textit{-algebra } \mathcal{D} \textit{ and} \\ &\quad a\ ^*\textit{-homomorphism } \rho : \mathcal{A} \to \mathcal{D},\ s.t.\ \rho(x) \neq 0. \\ \Longleftrightarrow\ &\mathcal{A} \textit{ embeds into } \prod_k D_k \textit{ for finite dimensional } C^*\textit{-algebras } \{\mathcal{D}_k\}_k \end{aligned}$$

RFD algebras include: finite dimensional C*-algebras, abelian C*-algebras. A result of Choi in 1980 showed that $C^*(F_2)$ is *RFD*.

Remark 2.2 *If a C*-algebra $\mathcal{A}$ is RFD, then $\mathcal{A}$ has a faithful trace.*

Theorem 2.3 (Malcev) *If G is a finite generated group, then*

$$C^*(G) \textit{ is } RFD \ \Rightarrow\ G \textit{ is } RF$$

Theorem 2.4 (Bekka, 2006) *$C^*(SL_4(\mathbb{Z}))$ is NOT RFD.*

Remark 2.5 *Connes embedding problem $\Longleftrightarrow$ is $C*(F_2 \times F_2)$ RFD*

In the context of C*-algebras, there are two types of free products we are interested: **full free product** and **reduced free product**.

Suppose that $\mathcal{A}_1$ and $\mathcal{A}_2$ are unital C*-algebras. The unital full free product, $\mathcal{D}(= \mathcal{A}_1 *_{\mathbb{C}} \mathcal{A}_2)$, is a unital C*-algebra together with unital $*$-homomorphism $\sigma_i : \mathcal{A}_i \to D$ such that the following is true:

> if $\mathcal{C}$ is a unital C*-algebra and $\rho_i : \mathcal{A}_i \to \mathcal{C}$ are unital $*$-homomorphisms, then $\exists$ a unique unital $*$-homomorphism $\pi : \mathcal{D} \to \mathcal{C}$ such that $\rho_i = \pi \circ \sigma_i$.

Reduced free products were introduced by Voiculescu in the context of free probability theory.

Theorem 2.6 (Exel, Loring, 1992) *Suppose that $\mathcal{A}_1$ and $\mathcal{A}_2$ are residually finite dimensional unital C*-algebras, then $\mathcal{A}_1 *_{\mathbb{C}I} \mathcal{A}_2$ is RFD, where $\mathcal{A}_1 *_{\mathbb{C}I} \mathcal{A}_2$ is the unital full free product of $\mathcal{A}_1$ and $\mathcal{A}_2$.*

How about amalgamated free products of RDF C*-algebras? Do we have an analogue of Baumslag's theorem in C*-algebra context?

3 An analogue of Baumslag's theorem

Suppose that unital C*-algebras $\mathcal{A} \supseteq \mathcal{D} \subseteq \mathcal{B}$. Using universal property, we can define the unital full amalgamated free product of $\mathcal{A}$ and $\mathcal{B}$ over $\mathcal{D}$, which is denoted by $\mathcal{A} *_{\mathcal{D}} \mathcal{B}$.

Remark 3.1 *If C*-algebra $\mathcal{A}$ is RFD, then $\mathcal{A}$ has a faithful trace.*

Example 3.2 *Let $\mathcal{D} = \mathbb{C} \oplus \mathbb{C}$, $\mathcal{A} = \mathcal{M}_2(\mathbb{C})$ and $\mathcal{B} = \mathcal{M}_3(\mathbb{C})$. Let $\mathcal{D} \hookrightarrow \mathcal{A}$ by sending $(a, b) \to diag(a, b)$ and $\mathcal{D} \hookrightarrow \mathcal{B}$ by sending $(a, b) \to diag(a, b, b)$. Then $\mathcal{A} *_{\mathcal{D}} \mathcal{B}$ is not RFD, because there is no trace on $\mathcal{A} *_{\mathcal{D}} \mathcal{B}$.*

An earlier result by S. Armstrong, K. Dykema, R. Exel, and H. Li in 2002:

Theorem 3.3 (Armstrong-Dykema-Exel-Li)
Suppose unital C-algebras: $\mathcal{A} \supseteq \mathcal{D} \subseteq \mathcal{B}$ with $\mathcal{A}$ and $\mathcal{B}$ finite dimensional. Then*

$\mathcal{A} *_{\mathcal{D}} \mathcal{B}$ is residually finite dimensional

$\iff \exists$ faithful tracial states $\tau_{\mathcal{A}}$ on $\mathcal{A}$ and $\tau_{\mathcal{B}}$ on $\mathcal{B}$ whose restrictions to $\mathcal{D}$ agree

$\iff \exists$ a matrix algbra $\mathcal{M}_k(\mathbb{C})$ and embedding $\rho_{\mathcal{A}} : \mathcal{A} \to \mathcal{M}_k(\mathbb{C})$, $\rho_{\mathcal{B}} : \mathcal{B} \to \mathcal{M}_k(\mathbb{C})$, such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{D} & \subseteq & \mathcal{A} \\ \cup & & \downarrow \rho_{\mathcal{A}} \\ \mathcal{B} & \underset{\rho_{\mathcal{B}}}{\to} & \mathcal{M}_k(\mathbb{C}) \end{array}$$

An analogue of Baumslag's Theorem in C*-algebra context by J. Shen and Q. Li in 2010:

Theorem 3.4 *Consider unital C*-algebras:* $\mathcal{A} \supseteq \mathcal{D} \subseteq \mathcal{B}$ *where* $\mathcal{A}$ *and* $\mathcal{B}$ *are RFD and* $\mathcal{D}$ *is finite dimensional. Then*

$\mathcal{A} *_{\mathcal{D}} \mathcal{B}$ *is residually finite dimensional*

$\Longleftrightarrow$ $\exists$ *a family of matrix algebras* $\{\mathcal{M}_{n_k}(\mathbb{C})\}$ *and embedding* $\rho_{\mathcal{A}} : \mathcal{A} \to \prod_k \mathcal{M}_{n_k}(\mathbb{C})$, $\rho_{\mathcal{B}} : \mathcal{B} \to \prod_k \mathcal{M}_{n_k}(\mathbb{C})$, *such that the following diagram commutes*

$$\begin{array}{ccc} \mathcal{D} & \subseteq & \mathcal{A} \\ \cup & & \downarrow \rho_{\mathcal{A}} \\ \mathcal{B} & \underset{\rho_{\mathcal{B}}}{\rightarrow} & \displaystyle\prod_k \mathcal{M}_{n_k}(\mathbb{C}) \end{array}$$

Corollary 3.5 *Consider unital C*-algebras:* $\mathcal{D} \subseteq \mathcal{A}$ *where* $\mathcal{A}$ *is RFD and* $\mathcal{D}$ *is finite dimensional. Then* $\mathcal{A} *_{\mathcal{D}} \mathcal{A}$ *is residually finite dimensional.*

There is another type of free product of C*-algebras: *Reduced free product* of C*-algebras introduced by D. Voiculescu.

Consider unital C*-algebras $\mathcal{A} \supseteq \mathcal{D} \subseteq \mathcal{B}$, $E_{\mathcal{A}} : \mathcal{A} \to \mathcal{D}$ and $E_{\mathcal{B}} : \mathcal{B} \to \mathcal{D}$ are condition expectations such that the corresponding GNS representations are faithful. Then, the reduced free product of $\mathcal{A}$ and $\mathcal{B}$ with the amalgamation over $\mathcal{D}$, denoted by $(\mathcal{A}, E_{\mathcal{A}}) *_{\mathcal{D}} (\mathcal{B}, E_{\mathcal{B}})$ is introduced by Voiculescu.

In particular, when $\mathcal{D} = \mathbb{C}$ and conditional expectations are induced by faithful traces, we obtained the reduced free product $(\mathcal{A}, \tau_{\mathcal{A}}) *_{red} (\mathcal{B}, \tau_{\mathcal{B}})$ of $\mathcal{A}$ and $\mathcal{B}$.

Most of reduced free products of unital C*-algebras are not RFD. For example,

$$C_r^*(F_2) \simeq (C_r^*(\mathbb{Z}), \tau_{\mathbb{Z}}) *_{red} (C_r^*(\mathbb{Z}), \tau_{\mathbb{Z}})$$

is not quasidiagonal by a result of Rosenberg, thus not RFD.

4 MF algebras

A separable C^*-algebras $\mathcal{A}$ is residually finite dimensional (RFD) $\Longleftrightarrow$ $\mathcal{A}$ embeds into $\prod_k D_k$ for a family of finite dimensional C^*-algebra $\{\mathcal{D}_k\}_k$.

MF algebras are introduced by Blackadar and Kirchberg in 1997.

Definition 4.1

$$\text{A separable } C^*\text{-algebras } \mathcal{A} \text{ is MF algebras (or } \mathcal{A} \text{ has MF property)}$$
$$\Longleftrightarrow \mathcal{A} \text{ embeds into } \prod_k D_k / \sum_k D_k \text{ for a family of matrix algebras } \{\mathcal{D}_k\}_k.$$

MF algebras include: RFD algebras, quasidiagonal C*-algebras.

Definition 4.2 *A separable C^*-algebra $\mathcal{A} \subseteq B(\mathcal{H})$ is quasidiagonal if there is an increasing sequence of finite-rank projections $\{p_i\}_{i=1}^{\infty}$ on H tending strongly to the identity such that $\|xp_i - p_ix\| \to 0$ as $i \to \infty$ for any $x \in \mathcal{A}$. An abstract separable C^*-algebra $\mathcal{A}$ is quasidiagonal if there is a faithful $*$-representation $\pi : \mathcal{A} \to B(\mathcal{H})$ on a Hilbert space $\mathcal{H}$ such that $\pi(\mathcal{A}) \subseteq B(\mathcal{H})$ is quasidiagonal.*

Proposition 4.3 (Blackadar, Kirchberg) *Suppose $\mathcal{A}$ is a nuclear C^*-algebra. Then*

$$\mathcal{A} \text{ is } MF \Leftrightarrow \mathcal{A} \text{ is quasidiagonal}$$

Applications of MF algebras:

Proposition 4.4 (Voicuelscu) *Suppose that $\mathcal{A}$ is an MF algebra, but not a quasidiagonal C^*-algebra. Then the BDF-extension semigroup, $Ext(\mathcal{A})$, is not a group.*

Proposition 4.5 (Voiculescu) *Suppose that $\mathcal{A}$ is an MF algebra. Then, for $x_1, \ldots, x_n$ in $\mathcal{A}$, we have*

$$\delta_{top}(x_1, \ldots, x_n) > -\infty,$$

where $\delta_{top}(x_1, \ldots, x_n)$ is Voiculescu's topological free entropy dimension of $x_1, \ldots, x_n$.

4.1 Reduced Free Products

Theorem 4.6 (Haagerup, Thorbjorsen)
$C_r^*(F_2) \simeq (C_r^*(\mathbb{Z}), \tau_{\mathbb{Z}}) *_{red} (C_r^*(\mathbb{Z}), \tau_{\mathbb{Z}})$ is an MF algebra. Thus $Ext(C_r^*(F_2))$ is not a group.

In 2009, D. Hadwin, J. Li, L. Wang and I showed that

Theorem 4.7 *Suppose that $\mathcal{A}_i$, $i = 1, \ldots, n$, is a family of unital separable AH algebras with faithful tracial states τ_i, $i = 1, \ldots, n$. Then*

$$(\mathcal{A}_1, \tau_1) *_{red} \cdots *_{red} (\mathcal{A}_n, \tau_n)$$

is an MF algebra.

Corollary 4.8 *Suppose that G_1, G_2 is direct products of abelian/finite groups. Then $C_r^*(G_1 * G_2)$ is an MF algebra. Moreover, if $|G_1| \geq 3$ and $|G_2| \geq 2$, then $Ext(C_r^*(G_1 * G_2))$ is not a group.*

4.2 Reduced Amalgamated Free Products

An extension of preceding results to reduced amalgamated free products is obtained by Q. Li and I in 2010.

Theorem 4.9 *Suppose that $\mathcal{A}_1$ and $\mathcal{A}_2$ are two unital UHF-algebras with faithful tracial states $\tau_{\mathcal{A}_1}$ and $\tau_{\mathcal{A}_2}$ respectively.*

Let $\mathcal{A}_1 \supseteq \mathcal{D} \subseteq \mathcal{A}_2$ be unital embedding of C^*-algebras where $\mathcal{D}$ is a finite-dimensional C^*-algebra.

Assume that $E_{\mathcal{A}_1} : \mathcal{A}_1 \to \mathcal{D}$ and $E_{\mathcal{A}_2} : \mathcal{A}_2 \to \mathcal{D}$ are the trace preserving conditional expectations from $\mathcal{A}_1$ and $\mathcal{A}_1$ onto $\mathcal{D}$ respectively.

Then the reduced amalgamated free product $(\mathcal{A}_1, E_{\mathcal{A}_1}) *_{\mathcal{D}} (\mathcal{A}_2, E_{\mathcal{A}_2})$ is an MF algebra if and only if $\tau_{\mathcal{A}_1}(z) = \tau_{\mathcal{A}_2}(z)$ for all $z \in \mathcal{D}$.

Corollary 4.10 *Suppose that $G_1 \supseteq H \subseteq G_2$ are finite groups. Then*

$$C_r^*(G_1 *_H G_2) \simeq C_r^*(G_1) *_{C_r^*(H)} C_r^*(G_2)$$

is an MF algebra. Moreover, if $[G_1 : H] \geq 2$ and $[G_2 : H] \geq 3$, then $Ext(C_r^*(G_1 *_H G_2))$ is not a group.

4.3 Full free products

In 2008, D. Hadwin, Q. Li and I showed that

Theorem 4.11 *Suppose that A_1 and $\mathcal{A}_2$ are unital MF algebras. Then the unital full free product of $\mathcal{A}_1$ and $\mathcal{A}_2$, $\mathcal{A}_1 *_{\mathbb{C}} \mathcal{A}_2$ is MF.*

Corollary 4.12 *Suppose that A_1 and $\mathcal{A}_2$ are unital MF algebras. Suppose that $x_1, \ldots, x_n$ and $y_1, \ldots, y_m$ are families of generators of $\mathcal{A}_1$, and $\mathcal{A}_2$ respectively. Then $x_1, \ldots, x_n, y_1, \ldots, y_m$ can be viewed as a family of generators of $\mathcal{A}_1 *_{\mathbb{C}} \mathcal{A}_2$. We have*

$$\delta_{top}(x_1, \ldots, x_n, y_1, \ldots, y_m) = \delta_{top}(x_1, \ldots, x_n) + \delta_{top}(y_1, \ldots, y_m),$$

where δ_{top} is Voiculescu's topological free entropy dimension for C^*-algebras.

4.4 Full Amalgamated free products

Example 4.13 *Let* $\mathcal{D} = \mathbb{C} \oplus \mathbb{C}$, $\mathcal{A} = \mathcal{M}_2(\mathbb{C})$ *and* $\mathcal{B} = \mathcal{M}_3(\mathbb{C})$. *Let* $\mathcal{D} \hookrightarrow \mathcal{A}$ *by sending* $(a, b) \to diag(a, b)$ *and* $\mathcal{D} \hookrightarrow \mathcal{B}$ *by sending* $(a, b) \to diag(a, b, b)$. *Then* $\mathcal{A}*_{\mathcal{D}}\mathcal{B}$ *is not MF, because there is no trace on* $\mathcal{A}*_{\mathcal{D}}\mathcal{B}$.

In 2010, Q. Li and I showed that

Theorem 4.14 *Consider unital C*-algebras:* $\mathcal{A} \supseteq \mathcal{D} \subseteq \mathcal{B}$ *where* $\mathcal{A}$ *and* $\mathcal{B}$ *are MF algebras and* $\mathcal{D}$ *is finite dimensional (or AF algebra, more generally). Then*

$\mathcal{A} *_{\mathcal{D}} \mathcal{B}$ *is MF algebra*

$\iff$ $\exists$ *a family of matrix algebras* $\{\mathcal{M}_{n_k}(\mathbb{C})\}$ *and embedding*

$$\rho_{\mathcal{A}} : \mathcal{A} \to \prod_k \mathcal{M}_{n_k}(\mathbb{C}) / \sum_k \mathcal{M}_{n_k}(\mathbb{C})$$

$$\rho_{\mathcal{B}} : \mathcal{B} \to \prod_k \mathcal{M}_{n_k}(\mathbb{C}) / \sum_k \mathcal{M}_{n_k}(\mathbb{C})$$

such that the following diagram commutes

$$\begin{array}{ccl} \mathcal{D} & \subseteq & \mathcal{A} \\ \cup & & \downarrow \rho_{\mathcal{A}} \\ \mathcal{B} & \underset{\rho_{\mathcal{B}}}{\to} & \prod_k \mathcal{M}_{n_k}(\mathbb{C}) / \sum_k \mathcal{M}_{n_k}(\mathbb{C}) \end{array}$$

Corollary 4.15 *Consider unital AF-algebras:* $\mathcal{A} \supseteq \mathcal{D} \subseteq \mathcal{B}$. *If there are faithful tracial states* $\tau_{\mathcal{A}}$ *and* $\tau_{\mathcal{B}}$ *on* $\mathcal{A}$ *and* $\mathcal{B}$ *respectively, such that* $\tau_{\mathcal{A}}(x) = \tau_{\mathcal{B}}(x)$ $\forall x \in \mathcal{D}$, *then* $\mathcal{A} *_{\mathcal{D}} \mathcal{B}$ *is an MF algebra.*