# A Noncommutative Gauss Map

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- We want to (i) consider some noncommutative extensions and (ii) study their dynamics.

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- We write $\alpha = [a_1, a_2, ...]$ in it's **continued fraction decomposition**.
- This provides a nice, algorithmic approximation of irrational numbers by rational numbers.

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$$e_n(x) := \left| m_n(x) - \frac{\ln(1+x)}{\ln 2} \right|$$

for large $n$.

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- (Wirsing 74) There is an optimal constant $q \sim .303$ such that $e_n(x) \leq q^n$.
- Why would Gauss say a good estimate would be "very desirable"?

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  $= m_n(\frac{1}{2}) - m_n(\frac{1}{3}) \sim \frac{\ln(9/8)}{\ln 2} + (.303..)^n.$

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- Step 1 in this program is importing the Gauss Map to the noncommutative world.
- We recall candidates for "noncommutative irrational numbers"
- And for the "noncommutative unit interval."

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- Let's think of $\mathcal{C}_\theta$ as a "noncommutative irrational number."

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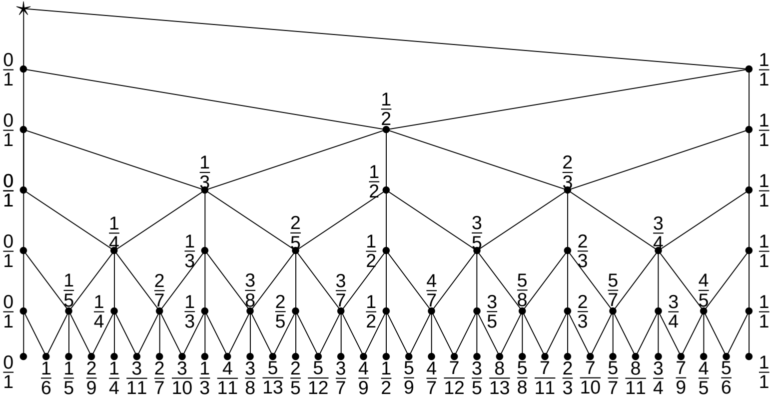


Figure: Bratelli diagram of $\mathfrak{A}$

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- So if we (incorrectly) view $\mathfrak{A}$ as operator valued continuous functions on its maximal ideal space..then each function evaluated at $\theta$ takes values in the Effros Shen algebra $\mathcal{C}_\theta$.
- For these reasons, we feel it is reasonable to attach the moniker "noncommutative unit interval" to $\mathfrak{A}$.

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- These are the properties we want our noncommutative extension to inherit.

# Gauss Measure on $\mathfrak{A}$

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**Theorem (E 10)**

*Let $\nu$ be a state on $C[0,1]$. Then $\nu$ has a unique tracial extension, $\tau_\nu$, to $\mathfrak{A}$.*

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We let $\tau_\mu$ be the unique tracial extension of $\mu$ to $\mathfrak{A}$.

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- Let's outline this extension for $s = 1$.

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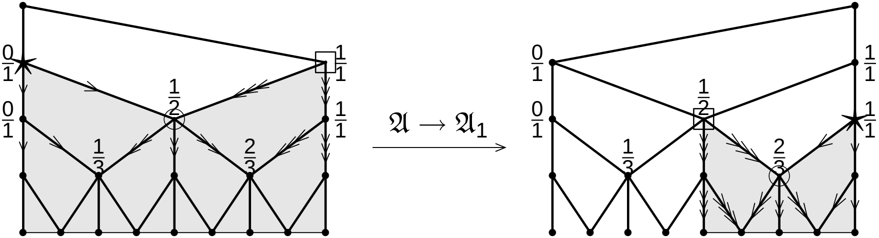


Figure: Bratelli Diagram of $\mathfrak{A}$.....Bratelli Diagram of Quotient of $\mathfrak{A}$

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- For this reason, we define a CP map that preserves as much trace as possible with induced map an $L^2$-isometry..

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$$T(x) = \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix} \quad \& \quad \phi_3(x) = (3/2)\tau_3\left( \begin{bmatrix} x_{11} & 0 \\ 0 & 0 \end{bmatrix} \right).$$

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$$\widetilde{\mathbb{G}} = \sum_{s=1}^{\infty} \widetilde{G}_s \frac{1+\theta}{(\theta+s)(\theta+s+1)}.$$

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4. $\widetilde{V}_G^* \pi_\phi(x) \widetilde{V}_G = \pi_{\tau_\mu}(\widetilde{\mathbb{G}}(x))$ *for* $x \in \mathfrak{A}$.

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4. $\widetilde{V}^*_G \pi_\phi(x) \widetilde{V}_G = \pi_{\tau_\mu}(\widetilde{\mathbb{G}}(x))$ *for* $x \in \mathfrak{A}$.
5. $\phi(x) = \tau_\mu(\widetilde{\mathbb{G}}(x))$ *for* $x \in \mathfrak{A}$.