Linear Programming and the Theory of Games

7

1 POLYHEDRAL CONVEX SETS

Recall that an equation containing one or more variables is called an *open statement.* For instance,

$$\text{(a)} \qquad -2x_1 + 3x_2 = 6$$

is an example of an open statement. If we let $A = (-2, 3)$, $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, and $b = 6$, we can write (a) in matrix form as

$$Ax = (-2, 3)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -2x_1 + 3x_2 = 6 = b.$$

For some two-component vectors x the statement $Ax = b$ is true and for others it is false. For instance, if $x = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$, it is true, since $-2 \cdot 3 + 3 \cdot 4 = 6$; and if $x = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$, it is false, since $-2 \cdot 2 + 3 \cdot 4 = 8$. The set of all two-component vectors x that make the open statement $Ax = b$ true is defined to be the *truth set* of the open statement.

EXAMPLE 1 In plane geometry it is usual to picture in the plane the truth sets of open statements such as (a). Thus we can regard each two-component vector x as being the components of a point in the plane in the usual way. Then the truth set or *locus* (which is the geometric term for truth set) of (a) is

For a nontechnical introduction to linear programming the reader should cover the first three sections; for a more technical exposition including the simplex method, cover the first six sections. For a nontechnical introduction to the theory of games, cover just Sections 8, 9, and 10; and for a technical introduction, cover the whole chapter.

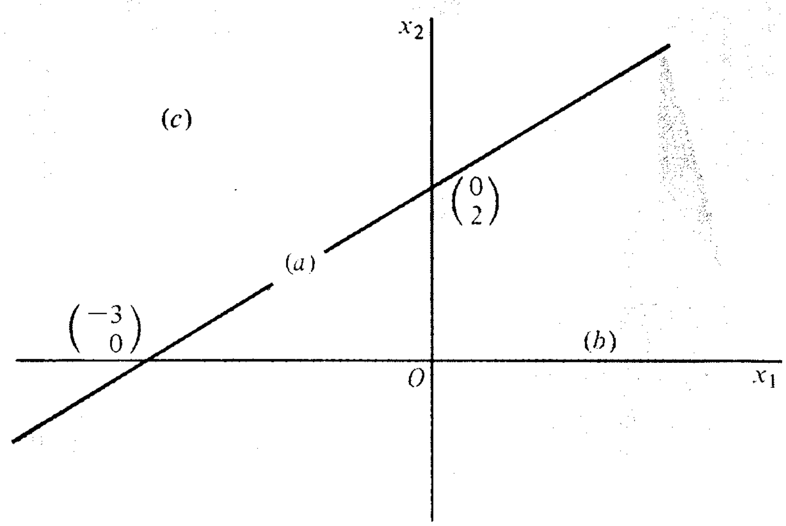


Figure 1

the straight line plotted in Figure 1. Points on this line may be obtained by assuming values for one of the variables and computing the corresponding values for the other variable. Thus, setting $x_1 = 0$, we find $x_2 = 2$, so that the point $x = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$ lies on the locus; similarly, setting $x_2 = 0$, we find $x_1 = -3$, so that the point $\begin{pmatrix} -3 \\ 0 \end{pmatrix}$ lies on the locus; and so on.

In the same way inequalities of the form $Ax \leq b$ or $Ax < b$ or $Ax \geq b$ or $Ax > b$ are open statements and possess truth sets. And in the case that x is a two-component vector, these can be plotted in the plane.

EXAMPLE 2 Consider the inequalities (b) $Ax < b$, (c) $Ax > b$, (d) $Ax \leq b$, and (e) $Ax \geq b$, where A, x, and b are as in Example 1. They may be written as

(b) $$-2x_1 + 3x_2 < 6,$$
(c) $$-2x_1 + 3x_2 > 6,$$
(d) $$-2x_1 + 3x_2 \leq 6,$$
(e) $$-2x_1 + 3x_2 \geq 6.$$

Consider (b) first. What points $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ satisfy this inequality? By trial and error we can find many points on the locus. Thus the point $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is on it, since $-2 \cdot 1 + 3 \cdot 2 = 4 < 6$; on the other hand, the point $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ is not on the locus, because $-2 \cdot 1 + 3 \cdot 3 = -2 + 9 = 7$, which is not less than 6.

In between these two points we find $\begin{pmatrix} 1 \\ \frac{8}{3} \end{pmatrix}$, which lies on the boundary—that is, on the locus of (a). We note that, starting with $\begin{pmatrix} 1 \\ \frac{8}{3} \end{pmatrix}$ on locus (a), by increasing x_2 we went outside the locus (b); by decreasing x_2 we came into the locus (b) again. This holds in general. Given a point on the locus of (a), by increasing its second coordinate we get more than 6, but by decreasing the second coordinate we get less than 6, and hence the latter gives a point in the truth set of (b). Thus we find that the locus of (b) consists of all points of the plane *below* the line (a)—in other words, the shaded area in Figure 1. The area on one side of a straight line is called an *open half-plane*.

We can apply exactly the same analysis to show that the locus of (c) is the open half-plane above the line (a). This can also be deduced from the fact that the truth sets of statements (a), (b), and (c) are disjoint and have as union the entire plane.

Since (d) is the disjunction of (a) and (b), the truth set of (d) is the union of the truth sets of (a) and (b). Such a set, which consists of an open half-plane together with the points on the line that define the half-plane, is called a *closed half-plane*. Obviously, the truth set of (e) consists of the union of (a) and (c) and therefore is also a closed half-plane.

Frequently we want to assert several different open statements at once—that is, we want to assert the conjunction of several such statements. The easy way to do this is to let A be an $m \times n$ matrix, x an n-component column vector, and b an m-component column vector. Then the statement $Ax \leq b$ is the conjunction of the m statements $A_i x \leq b_i$, where A_i *is the* ith row of A and b_i is the ith entry of b.

EXAMPLE 3 A box manufacturer makes small and large boxes from a single kind of cardboard. The small boxes require 2 square feet of cardboard each and the large boxes 3 square feet each. If the manufacturer has 60 square feet of cardboard on hand, what are the possible combinations of small and large boxes that he can make?

In order to set up this problem let x_1 be the number of small boxes and x_2 the number of large boxes to be made. Since it is impossible to make negative numbers of boxes, we have the obvious constraints

(f) $$x_1 \geq 0,$$

(g) $$x_2 \geq 0.$$

Also, because of the constraint on the total amount of cardboard on hand, we have

(h) $$2x_1 + 3x_2 \leq 60.$$

If we now want to state these three inequality constraints simultaneously in the form $Ax \leq b$, we must first change (f) and (g) into $\leq$ constraints.

This can be done by multiplying through by -1, so that (f) becomes $-x_1 \leq 0$ and (g) becomes $-x_2 \leq 0$. If we now define

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 2 & 3 \end{pmatrix}, \qquad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad b = \begin{pmatrix} 0 \\ 0 \\ 60 \end{pmatrix},$$

we see that $Ax \leq b$ is a matrix way of asserting the conjunction of (f), (g), and (h). The truth set of $Ax \leq b$ is the intersection of the three individual truth sets. The truth set of (f) is the right half-plane; the truth set of (g) is the upper half-plane; and the truth set of (h) is the half-plane below and on the line $2x_1 + 3x_2 = 60$. The intersection of these is the triangle (including the sides and corners) shaded in Figure 2. The area shaded in Figure 2 contains all those and only those points that simultaneously satisfy (f), (g), and (h), or, equivalently, $Ax \leq b$.

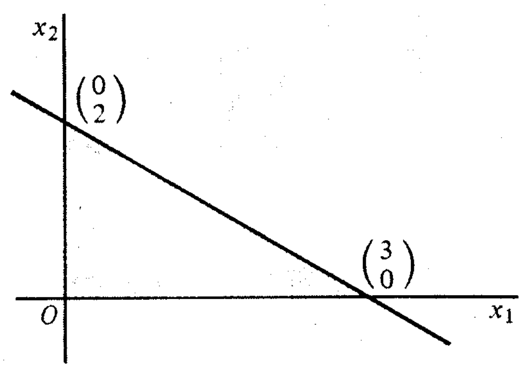


Figure 2

In the examples considered so far we have restricted ourselves to open statements with two variables. Such statements have truth sets that can be sketched in the plane. In the same way, open statements with three variables have truth sets that can be visualized in three-dimensional space. Open statements with four or more variables have truth sets in four or more dimensions, which we can no longer visualize. However, applied problems frequently lead to such statements. Fortunately, we shall develop methods (in Section 5) for handling them without having to visualize the truth sets geometrically.

In order to have a notation that will enable us to talk in general about conjunctions of several open statements in any number of dimensions, we shall, for the remainder of this chapter, consider b to be an m-component column vector, x an n-component column vector, and A an $m \times n$ matrix. The ith row of A will be denoted by A_i. Similarly, the ith component of b will be denoted by b_i. Of course, A_i is an n-component row vector and b_i is a number. We shall let $\mathfrak{X}_n$ denote the set of all n-component column vectors x. Thus in Example 3 we had $m = 3$ and $n = 2$. A was a 3×2

matrix, x a two-component column vector, and b a three-component column vector. The set of all two-component column vectors x is denoted by $\mathfrak{X}_2$.

We now set up some definitions that will be used in the later exposition.

Definition The truth set of $A_i x = b_i$ is called a *hyperplane* in $\mathfrak{X}_n$. The truth sets of inequalities of the form $A_i x < b_i$ or $A_i x > b_i$ are called *open half-spaces*, while the truth sets of the inequalities $A_i x \leq b_i$ or $A_i x \geq b_i$ are called *closed half-spaces* in $\mathfrak{X}_n$.

When we assert the conjunction of several open statements, the resulting truth set is the intersection of the truth sets of the individual open statements. Thus in Example 3 we have the conjunction of $m = 3$ open statements in $\mathfrak{X}_2$. In Figure 2 we show this geometrically as the intersection of $m = 3$ closed half-spaces (-planes) in $n = 2$ dimensions. Such intersections of closed half-spaces are of special importance.

Definition The intersection of a finite number of closed half-spaces is a *polyhedral convex set.*

Theorem Any polyhedral convex set is the truth set of an inequality statement of the form $Ax \leq b$.

Proof A closed half-space is the truth set of an inequality of the form $A_i x \leq b_i$. (An inequality of the form $A_i x \geq b_i$ can be converted into one of this form by multiplying by -1.) Now a polyhedral convex set is the truth set of the conjunction of several such statements. Since A is the matrix whose ith row is A_i and b is the column vector with components b_i, then the inequality statement $Ax \leq b$ is a succinct way of stating the conjunction of the inequalities $A_1 x \leq b_1, \ldots, A_m x \leq b_m$. This completes the proof.

The terminology polyhedral *convex* sets is used because these sets are special examples of convex sets. A convex set C is a set such that whenever u and v are points of C, the entire line segment between u and v also belongs to C. This is equivalent to saying that all points of the form $z = au + (1 - a)v$ for $0 \leq a \leq 1$ belong to C whenever u and v do. In this chapter we shall be concerned primarily with polyhedral convex sets.

EXERCISES

1. Draw pictures of the truth sets of $Ax \leq b$, where A and b are as given below. (Construct the truth sets of the individual statements first and then take their intersection.)

(a) $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & -3 \end{pmatrix}, \quad b = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}.$

(b) $A = \begin{pmatrix} -2 & -3 \\ -1 & 1 \\ 1 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} -6 \\ 2 \\ 3 \end{pmatrix}.$

(c) $A = \begin{pmatrix} 2 & 3 \\ -1 & 1 \\ 1 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 6 \\ 2 \\ 3 \end{pmatrix}.$

(d) $A = \begin{pmatrix} 0 & -1 \\ -1 & 0 \\ 1 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}.$

(e) $A = \begin{pmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 2 \\ 2 \\ 3 \\ 3 \end{pmatrix}.$

(f) $A = \begin{pmatrix} 3 & 2 \\ 3 & 2 \end{pmatrix}, \quad b = \begin{pmatrix} -6 \\ 6 \end{pmatrix}.$

(g) $A = \begin{pmatrix} -3 & -2 \\ 3 & 2 \end{pmatrix}, \quad b = \begin{pmatrix} -6 \\ 6 \end{pmatrix}.$

(h) $A = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$

(i) $A = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 2 \\ -5 \end{pmatrix}.$

(j) $A = \begin{pmatrix} -3 & -2 \\ -2 & -3 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} -6 \\ -6 \\ 0 \\ 0 \end{pmatrix}.$

(k) $A = \begin{pmatrix} -2 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} -7 \\ 0 \\ 0 \end{pmatrix}.$

2. In the cardboard-box problem of Example 3 consider the following additional constraints:
   (a) "At least as many small as large boxes should be made." Write a constraint involving x_1 and x_2 that expresses this and find A and b. Draw the picture of the resulting convex set. [*Partial ans.* $-x_1 + x_2 \leq 0$.]
   (b) In addition to the constraints above add a constraint expressing; "at most 20 small boxes should be made." Find A and b and sketch the convex set. [*Partial ans.* $x_1 \leq 20$.]
3. Of the polyhedral convex sets constructed in Exercise 1, which have a finite area and which have infinite area?
   [*Partial ans.* (c), (d), (f), (h), and (j) are of infinite area; (g) is a line; (i) and (k) are empty.]

4. For each of the following half-planes give an inequality of which it is the truth set.
   (a) The open half-plane above the x_1 axis. [*Ans.* $x_2 > 0$.]
   (b) The closed half-plane on and above the straight line making angles of 45 degrees with the positive x_1 and x_2 axis.

Exercises 5 through 9 refer to a situation in which a retailer is trying to decide how many units of items X and Y he should keep in stock. Let x be the number of units of X and y be the number of units of Y. X costs \$4 per unit and Y costs \$3 per unit.

5. One cannot stock a negative number of units of either X or Y. Write these conditions as inequalities and draw their truth sets.
6. The maximum demand over the period for which the retailer is contemplating holding inventory will not exceed 600 units of X or 600 units of Y. Modify the set found in Exercise 5 to take this into account.
7. The retailer is not willing to tie up more than \$2400 in inventory altogether. Modify the set found in Exercise 6.
8. The retailer decides to invest at least twice as much in inventory of item X as he does in inventory of item Y. Modify the set of Exercise 7.
9. Finally, the retailer decides that he wants to invest \$900 in inventory of item Y. What possibilities are left? [*Ans.* None.]

10. Assume that a pound of meat contains 80 units of protein and 10 units of calcium while a quart of milk contains 15 units of protein and 60 units of calcium. If an adult's minimum daily requirements are 40 units of protein and 30 units of calcium, what consumption quantities of meat and milk will yield at least these minimum daily requirements? A convenient way to summarize the data is by the following *data box:*

Food	Protein	Calcium
Meat	$80 \dfrac{\text{units protein}}{\text{lb meat}}$	$10 \dfrac{\text{units calcium}}{\text{lb meat}}$
Milk	$15 \dfrac{\text{units protein}}{\text{qt milk}}$	$60 \dfrac{\text{units calcium}}{\text{qt milk}}$
Requirements	$40 \dfrac{\text{units protein}}{\text{day}}$	$30 \dfrac{\text{units calcium}}{\text{day}}$

   (a) Let w_1 be the number of pounds of meat and w_2 be the number of quarts of milk consumed per day, and let $w = (w_1, w_2)$. Write inequality constraints that will solve the above problem. Find A and c so that they can be written $wA \geq c$.

(b) Sketch the set of feasible vectors. Show that it is unbounded (that it has infinite area).
(c) Show that another way of indicating units is as in the data box that follows:

	Protein	Calcium	
Meat	80	10	(per pound)
Milk	15	60	(per quart)
Requirements	40	30	(per day)
	(units)	(units)	

2 EXTREME POINTS; MAXIMA AND MINIMA OF LINEAR FUNCTIONS

In the present section we first discuss the problem of finding the extreme points of a bounded convex polyhedral set. Then we find out how to compute the maximum and minimum values of a linear function defined on such a set.

We use the following notation: the polyhedral convex set C is the truth set of the statement $Ax \leq b$, where A is an $m \times n$ matrix, x is an n-component column vector, and b is an m-component column vector. We let $A_1, A_2, \ldots, A_m$ denote the rows of A. Hence A_i is an n-component row vector and

$$A = \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{pmatrix}.$$

The statement $Ax \leq b$ is then the conjunction of the statements

$$A_1x \leq b_1, \quad A_2x \leq b_2, \quad \ldots, \quad A_mx \leq b_m.$$

Definition We shall call the truth set of the statement $A_ix = b_i$ the bounding hyperplane of the half space $A_ix \leq b_i$.

Thus, in Figure 1 of the preceding section the slanting line (a) is the bounding hyperplane of the half-space (b).

We found in the previous section that a convex set C is the intersection of a finite number of half-spaces. The bounding hyperplanes of these half-spaces that also contain points of C are called *bounding hyperplanes* of C. Thus in Example 3 of Section 1 the bounding hyperplanes of the polyhedral convex set given there are the three boundary lines of the triangle shaded in Figure 2. Note that these lines intersect in pairs in three points, the vertices of the triangle.

Definition Let C be the polyhedral convex set defined by $Ax \leq b$, where x is an n-component vector. Then a point T is an *extreme* (or *corner*) *point* of C if it

a. belongs to C, and
b. is the intersection of n bounding hyperplanes of C.

EXAMPLE 1 Find the extreme points of the polyhedral convex set $Ax \leq b$, where

$$A = \begin{pmatrix} 2 & 3 \\ -2 & -1 \\ 0 & -1 \end{pmatrix}, \qquad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad b = \begin{pmatrix} 60 \\ -32 \\ -2 \end{pmatrix}.$$

The corresponding inequalities are:

$$\begin{aligned} 2x_1 + 3x_2 &\leq 60, \\ 2x_1 + x_2 &\geq 32, \\ x_2 &\geq 2. \end{aligned}$$

The last two inequalities have been multiplied through by -1, and can be regarded as managerial constraints added to the box-manufacturer problem of Example 3 of Section 1. A sketch of the three half-planes (Figure 3)

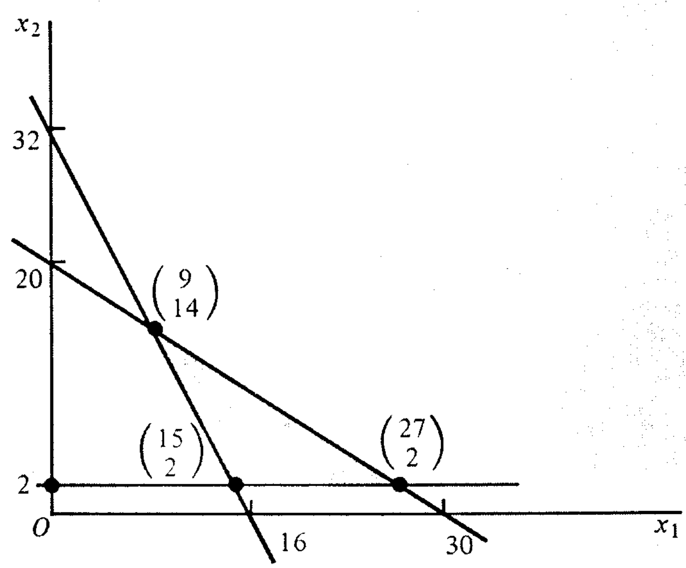


Figure 3

shows that the set of feasible solutions is a triangle. Hence we can find the extreme points by changing the inequalities to equalities in pairs and solving three sets of simultaneous equations. We obtain in this way the points

$$\begin{pmatrix} 9 \\ 14 \end{pmatrix}, \quad \begin{pmatrix} 15 \\ 2 \end{pmatrix}, \quad \begin{pmatrix} 27 \\ 2 \end{pmatrix},$$

which are the extreme points of the set.

We can now give an interpretation for the various points of the polyhedral convex set in terms of the system of inequalities. An extreme point lines on two boundaries, which means that two of the inequalities are actually equalities. A point on a side, other than an extreme point, lies on one boundary and hence one inequality is an equality. An interior point of the polygon must, by a process of elimination, correspond to the case where the inequalities are all strict inequalities—that is, not only $\leq$ but $<$ holds.

There is a mechanical (but lengthy) method for finding all the extreme points of a polyhedral convex set C defined by $Ax \leq b$. Consider the bounding hyperplanes $A_1x = b_1, \ldots, A_mx = b_m$ of the half-spaces that determine C. Select a subset of n of these hyperplanes and solve their equations simultaneously. If the result is a unique point x^0, then (and only then) check to see whether x^0 belongs to C. If it does, by the above definition, x^0 is an extreme point of C. Moreover, all extreme points of C can be found in this manner.

EXAMPLE 2 Let

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Then the polyhedral convex set C defined by $Ax \leq b$ is the first quadrant of the x_1, x_2 plane, shaded in Figure 4. The only extreme point is the origin, which is the intersection of the lines $x_1 = 0$ and $x_2 = 0$. This is an example of an *unbounded* polyhedral convex set.

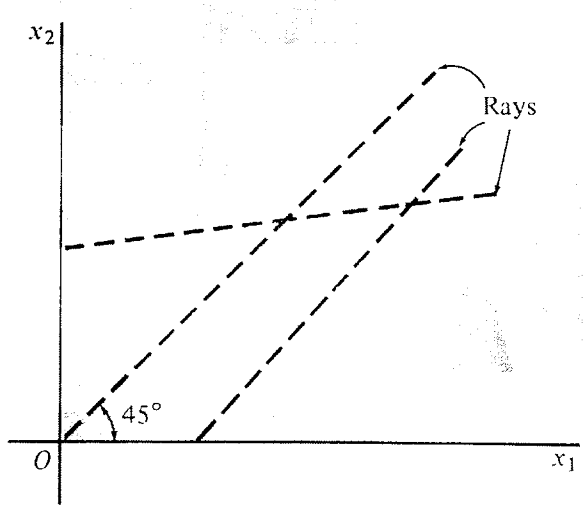


Figure 4

Notice that the set C in Example 2 contains the *ray* or half-line that starts at the origin of coordinates and extends upward to the right making a 45-degree angle with the axes. This ray is dotted in Figure 4. Of course, this set also contains many other rays; two others are shown in the figure.

We shall say that a polyhedral convex set is *bounded* if it does not contain a ray. A set, such as the one in Figure 4, that does contain rays will be called *unbounded*. For simplicity we shall restrict our discussion in most of this chapter to bounded convex sets.

EXAMPLE 3 Consider the box-manufacturer problem of Example 3 of Section 1, and suppose that the manufacturer makes a profit of \$1 on small and \$2 on large boxes. Hence, if he makes x_1 small and x_2 large boxes, his profit function is $x_1 + 2x_2$, and the inequalities limiting the choice of x_1 and x_2 are given in Example 1. What is the most and what the least profit he can make?

We must find the maximum and the minimum value of $x_1 + 2x_2$ for point (x_1, x_2) in the triangle shaded in Figure 3. Let us first try the extreme points. At $(15, 2)$ we have a profit of 19, at $(27, 2)$ a profit of 31, and at $(9, 14)$ a profit of 37. The last extreme point is most profitable. But what can we say about the remainder of the triangle? If we start at $(9, 14)$ and try to move to other points in the triangle, the best thing to do is to move along the bounding hyperplane $2x_1 + 3x_2 = 60$, since in this way we can get the most favorable tradeoff between x_1 and x_2. However, for each unit we decrease x_2 along this line we can increase x_1 by only $\frac{3}{2}$ units, with a net loss of profit. Hence the maximum profit is taken on at the extreme point $(9, 14)$. A similar argument shows that the minimum profit is taken on at the extreme point $(15, 2)$. Thus for this example the maximum and minimum profits are observed at extreme points. We shall show that this is true in general.

Given a convex polyhedral set C and a linear function

$$cx = c_1x_1 + c_2x_2 + \dots + c_nx_n,$$

where $c = (c_1, c_2, \dots, c_n)$, we want to show in general that the maximum and minimum values of the function cx always occur at extreme points of C. We shall carry out the proof for the planar case in which $n = 2$, but our results are true in general.

First, we shall show that the values of the linear function $c_1x_1 + c_2x_2$ on any line segment lie *between* the values the function has at the two endpoints (possibly equal to the value at one endpoint). We represent the points as column vectors $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and then we see that our linear function is represented by the row vector (c_1, c_2). Let the endpoints of the segment be

$$p = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} \quad \text{and} \quad q = \begin{pmatrix} x_1'' \\ x_2'' \end{pmatrix}.$$

We have seen in Chapter 4 (see Figure 4) that the points in between p and q can be represented as $tp + (1 - t)q$, with $0 \leq t \leq 1$. If the values of the function at the points p and q are P and Q, respectively (assume that $P \geq Q$), then at a point in between the value will be $tP + (1 - t)Q$, since the function is linear. This value can also be written as

$$tP + (1 - t)Q = Q + (P - Q)t,$$

which (for $0 \leq t \leq 1$) is at least Q and at most P.

We are now in a position to prove the result illustrated in Example 3.

Theorem A linear function cx defined over a convex polyhedral set C takes on its maximum (and minimum) value at an extreme point of C.

Proof The proof of the theorem is illustrated in Figure 5. We shall suppose that at the extreme point p the function takes on a value P greater than or equal to the value at any other extreme point, and at the extreme point

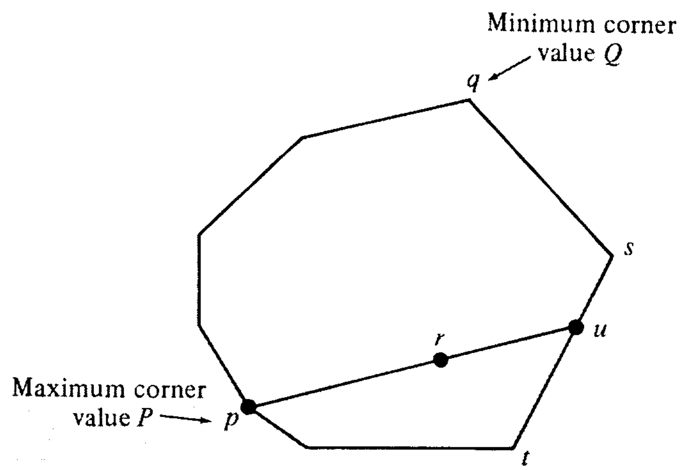


Figure 5

q it takes on its smallest extreme-point value, Q. Let r be any point of the polygon. Draw a straight line between p and r and continue it until it cuts the polygon again at a point u lying on an edge of the polygon, say the edge between the corner points s and t. (The line may even cut the edge at one of the points s and t; the analysis remains unchanged.) By hypothesis the value of the function at any corner point must lie between Q and P. By the above result the value of the function at u must lie between its values at s and t, and hence must also lie between Q and P. Again by the above result the value of the function at r must lie between its values at p and u, and hence must also lie between Q and P. Since r was any point of the polygon, our theorem is proved.

Suppose that in place of the linear function $c_1x_1 + c_2x_2$ we had considered the function $c_1x_1 + c_2x_2 + k$. The addition of the constant k merely changes every value of the function, including the maximum and minimum values of the function, by that amount. Hence the analysis of where the maximum and minimum values of the function are taken on is unchanged. Therefore, we have the following theorem.

Theorem The function $cx + k$ defined over a convex polyhedral set C takes on its maximum (and minimum) value at an extreme point of C.

A method of finding the maximum or minimum of the function $cx + k$ defined over a convex set C is then the following: Find the extreme points of the set; there will be a finite number of them; substitute the coordinates of each into the function; the largest of the values so obtained will be the maximum of the function and the smallest value will be the minimum of the function. The method was illustrated previously in Example 3.

In Section 5 we shall describe the so-called *simplex method,* which is considerably more efficient for solving the problem in Example 3.

EXERCISES

1. Consider the cardboard-box problem of Exercise 2 of Section 1. Assuming that both constraints stated in (a) and (b) are in effect and the profit function is $x_1 + 2x_2$, find the extreme point (or points) that give maximum and minimum profit.
2. Rework Exercise 1 with profit function $2x_1 + 3x_2$. Show that in this case there is more than one solution for maximum profit.
3. Consider the diet problem of Exercise 10 of Section 1. Suppose that meat costs \$1 per pound and milk costs 30 cents per quart. Find the lowest-cost diet that will meet minimum requirements.
   [*Ans.* $w = (\frac{13}{31}, \frac{40}{93})$, cost is $\$\frac{17}{31}$.]
4. The owner of an oil truck with a capacity of 500 gallons hauls gasoline and oil products from city to city. On any given trip he wishes to load his truck with at least 200 gallons of regular gasoline, at least 100 gallons of high-test gasoline, and at most 150 gallons of kerosene. Assuming that he always fills his truck to capacity, find the convex set of ways that he can load his truck. Interpret the extreme points of the set. [*Hint:* There are four extreme points.]
5. An advertiser wishes to sponsor a half-hour television comedy and must decide on the composition of the show. The advertiser insists that there be at least three minutes of commercials, while the television network requires that the commercial time be limited to at most 15 minutes. The comedian refuses to work more than 22 minutes each half-hour show. If a band is added to play while neither the comedian nor the commercials are on, construct the convex set C of possible assignments of time to the comedian, the commercials, and the band that use up the 30 minutes. Find the extreme points of C.
   [*Ans.* if x_1 is the comedian time, x_2 the commercial time, and $30 - x_1 - x_2$ the band time, the extreme points are
   $$\begin{pmatrix}0\\3\end{pmatrix}, \begin{pmatrix}22\\3\end{pmatrix}, \begin{pmatrix}22\\8\end{pmatrix}, \begin{pmatrix}15\\15\end{pmatrix} \text{ and } \begin{pmatrix}0\\15\end{pmatrix}.]$$

6. In Exercise 4 suppose that the oil truck operator gets 3 cents per gallon for delivering regular gasoline, 2 cents per gallon for high-test, and 1 cent per gallon for kerosene. Write the expression that gives the total amount he will get paid for each possible load that he carries. How should he load his truck in order to earn the maximum amount? [*Ans.* He should carry 400 gallons of regular gasoline, 100 gallons of high test, and no kerosene.]
7. In Exercise 6, if he gets 3 cents per gallon for regular and 2 cents per gallon for high-test gasoline, how high must his payment for kerosene become before he will load it on his truck in order to make a maximum profit?

   [*Ans.* He must get paid at least 3 cents per gallon of kerosene.]
8. In Exercise 5 let x_1 be the number of minutes the comedian is on and x_2 the number of minutes the commercial is on the program. Suppose the comedian costs \$200 per minute, the commercials cost \$50 per minute, and the band is free. How should the advertiser choose the composition of the show in order that its costs be a minimum?
9. Consider the polyhedral convex set P defined by the inequalities

$$-1 \leq x_1 \leq 4,$$
$$0 \leq x_2 \leq 6.$$

   Find four different sets of conditions on the constants a and b that the function $\mathbf{F}(x) = ax_1 + bx_2$ should have its maximum at one and only one of the four corner points of P. Find conditions that $\mathbf{F}$ should have its minimum at each of these points.

   [*Ans.* For example, the maximum is at $\begin{pmatrix} 4 \\ 6 \end{pmatrix}$ if $a > 0$ and $b > 0$.]
10. A well-known nursery rhyme goes, "Jack Sprat could eat no fat, his wife could eat no lean. . . ." Suppose Jack wishes to have at least one pound of lean meat per day, while his wife (call her Jill) needs at least .4 pound of fat per day. Assume they buy only beef having 10 percent fat and 90 percent lean, and pork having 40 percent fat and 60 percent lean. Jack and Jill want to fulfill their minimal diet requirements at the lowest possible cost.
    (a) Let x be the amount of beef and y the amount of pork they purchase per day. Construct the convex set of points in the plane representing purchases that fulfill both persons' minimum diet requirements.
    (b) Suggest necessary restrictions on the purchases that will change this set into a convex polygon.
    (c) If beef costs \$1 per pound, and pork costs 50 cents per pound, show that the diet of least cost has only pork, and find the minimum cost. [*Ans.* \$.83.]
    (d) If beef costs 75 cents and pork costs 50 cents per pound, show that there is a whole-line segment of solution points and find the minimum cost. [*Ans.* \$.83.]

(e) If beef and pork each cost \$1 a pound, show that the unique minimal cost diet has both beef and pork. Find the minimum cost. [*Ans.* \$1.40.]

(f) Show that the restriction made in part (b) did not alter the answers given in (c)–(e).

11. In Exercise 10(d) show that for all but one of the minimal-cost diets Jill has more than her minimum requirement of fat, while Jack always gets exactly his minimal requirement of lean. Show that all but one of the minimal-cost diets contains some beef.

12. In Exercise 10(e) show that Jack and Jill each get exactly their minimal requirements.

13. In Exercise 10 if the price of pork is fixed at \$1 a pound, how low must the price of beef fall before Jack and Jill will eat only beef? [*Ans.* \$.25.]

14. In Exercise 10 suppose that Jack decides to reduce his minimal requirement to .6 pound of lean meat per day. How does the convex set change? How do the solutions in 3(c), (d), and (e) change?

3 LINEAR PROGRAMMING PROBLEMS

An important class of practical problems are those that require the determination of the maximum or the minimum of a linear function $cx + k$ defined over a polyhedral convex set of points C. We illustrate these so-called *linear programming problems* by means of the following examples. In Section 5 we shall discuss the simplex method for solving these examples.

EXAMPLE 1 An automobile manufacturer makes automobiles and trucks in a factory that is divided into two shops. Shop 1, which performs the basic assembly operation, must work 5 man-days on each truck but only 2 man-hours on each automobile. Shop 2, which performs finishing operations, must work 3 man-hours for each automobile or truck that it produces. Because of men and machine limitations shop 1 has 180 man-hours per week available while shop 2 has 135 man-hours per week. If the manufacturer makes a profit of \$300 on each truck and \$200 on each automobile, how many of each should he produce to maximize his profit?

Before proceeding, let us summarize the problem in the *data box* of Figure 6. (The term *data box* is due to A. W. Tucker.) Notice that the numbers introduced above appear in the data box with their physical dimensions attached. When doing dimensional analysis, in the sense of physics, we may manipulate these dimension quantities just like algebraic quantities. We shall see in Section 6 that we can obtain interpretations for dual variables by means of dimensional analysis. The reader is strongly advised to set up a similar data box for every linear programming example he works.

An alternate and slightly more elegant way of indicating the units in the data box is shown in Figure 7. The reader should compare it with Figure

	Trucks	Autos	Capacities
Shop 1	$5\ \frac{\text{S1-manhr}}{\text{truck}}$	$2\ \frac{\text{S1-manhr}}{\text{auto}}$	$180\ \frac{\text{S1-manhr}}{\text{week}}$
Shop 2	$3\ \frac{\text{S2-manhr}}{\text{truck}}$	$3\ \frac{\text{S2-manhr}}{\text{auto}}$	$135\ \frac{\text{S2-manhr}}{\text{week}}$
Profits	$300\ \frac{\$}{\text{truck}}$	$200\ \frac{\$}{\text{auto}}$	

Figure 6

	Trucks	Autos	Capacities	
Shop 1	5	2	180	(Man-hours)
Shop 2	3	3	135	(Man-hours)
Profits	300	200		($)
	(per truck)	(per auto)	(per week)	

Figure 7

6 to see the correspondence between them. When in doubt, use the more explicit indications of Figure 6.

A dimensional fraction such as "S1-manhr/truck" is read "shop 1 man-hours per truck." Suppose we now introduce two variables x_1 with dimensions "trucks/week," which will become the number of trucks per week we should produce, and x_2 with dimensions "autos/week." Then the first constraint of the data box of Figure 6 becomes:

$$\left(5\,\frac{\text{S1-manhr}}{\text{truck}}\right)\left(x_1\,\frac{\text{trucks}}{\text{week}}\right) + \left(2\,\frac{\text{S1-manhr}}{\text{auto}}\right)\left(x_2\,\frac{\text{autos}}{\text{week}}\right) \le 180\,\frac{\text{S1-manhr}}{\text{week}}.$$

Now, by canceling the common term "truck" from numerator and denominator of the first term, and similarly canceling the common dimension "auto" from the numerator and denominator of the second term, we see that the resulting dimensions of each term are "S1-manhr/week"—the same as the dimensions of the capacity term on the right-hand side of the inequality. A similar dimensional analysis can be carried out for the second capacity constraint. Dropping dimensions, we have the following restrictions:

$$5x_1 + 2x_2 \le 180,$$
$$3x_1 + 3x_2 \le 135,$$

together with the obviously necessary nonnegative constraints $x_1 \ge 0$ and $x_2 \ge 0$.

Subject to these constraints we want to maximize the profit function:

$$\left(300\,\frac{\$}{\text{truck}}\right)\left(x_1\,\frac{\text{trucks}}{\text{week}}\right) + \left(200\,\frac{\$}{\text{auto}}\right)\left(x_2\,\frac{\text{autos}}{\text{week}}\right).$$

Canceling out the common terms, we see that the dimensions of this function are simply "$/week."

In order to state the problem as a linear programming problem we define the quantities:

$$A = \begin{pmatrix} 5 & 2 \\ 3 & 3 \end{pmatrix}, \qquad b = \begin{pmatrix} 180 \\ 135 \end{pmatrix}, \quad \text{and} \quad c = (300, 200),$$

which are immediately evident from the data boxes in Figure 6 and 7. Then our problem is:

Maximum problem: Determine the vector $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ so that the weekly profit, given by the quantity cx, is a maximum subject to the inequality constraints $Ax \leq b$ and $x \geq 0$. The inequality constraints insure that the weekly number of available man-hours is not exceeded and that nonnegative quantities of automobiles and trucks are produced.

The graph of the convex set of possible x vectors is pictured in Figure 8. This is a problem of the kind discussed in the previous section.

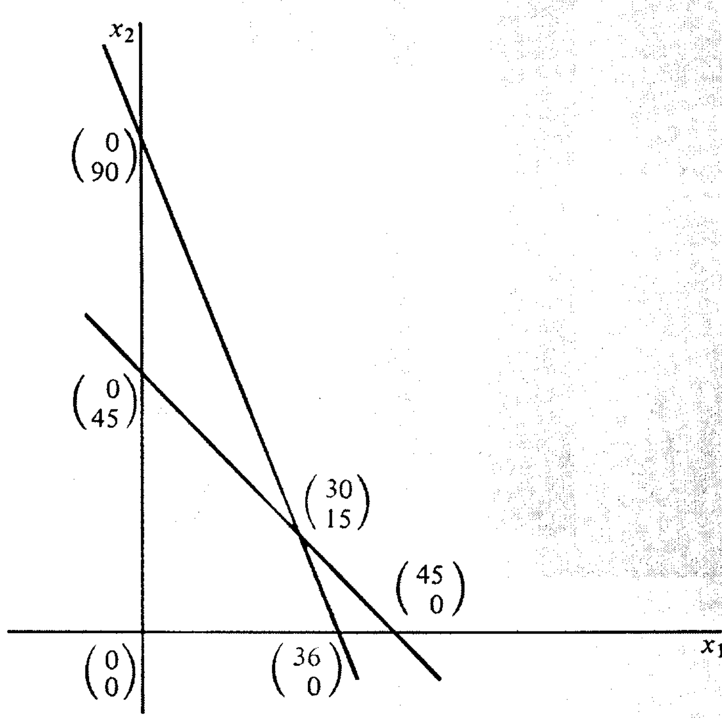


Figure 8

The extreme points of the convex set C are

$$T_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad T_2 = \begin{pmatrix} 36 \\ 0 \end{pmatrix}, \qquad T_3 = \begin{pmatrix} 0 \\ 45 \end{pmatrix}, \quad \text{and} \quad T_4 = \begin{pmatrix} 30 \\ 15 \end{pmatrix}.$$

Following the solution procedure outlined in the previous section, we test the function $cx = 300x_1 + 200x_2$ at each of these extreme points. The values taken on are 0, 10,800, 9000, and 12,000. Thus the maximum weekly profit is $12,000, achieved by producing 30 trucks and 15 automobiles per week.

EXAMPLE 2 A mining company owns two different mines that produce a given kind of ore. The mines are located in different parts of the country and have different production capacities. After crushing, the ore is graded into three classes: high-grade, medium-grade, and low-grade ores. There is some demand for each grade of ore. The mining company has contracted to provide a smelting plant with 12 tons of high-grade, 8 tons of medium-grade, and 24 tons of low-grade ore per week. It costs the company $200 per day to run the first mine and $160 per day to run the second. However, in a day's operation the first mine produces 6 tons of high-grade, 2 tons of medium-grade, and 4 tons of low-grade ore, while the second mine produces daily 2 tons of high-grade, 2 tons of medium-grade, and 12 tons of low-grade ore. How many days a week should each mine be operated in order to fulfill the company's orders most economically?

Before proceeding, we again summarize the problem in the data boxes of Figures 9 and 10. The reader should compare these two figures to see the correspondence between them. We shall make use of these dimensions when we give interpretations of the dual variables in Section 6.

	High-grade Ore HG	Medium-grade Ore MG	Low-grade Ore LG	Cost
Mine 1	$6\ \dfrac{\text{tons-HG}}{\text{M1-day}}$	$2\ \dfrac{\text{tons-MG}}{\text{M1-day}}$	$4\ \dfrac{\text{tons-LG}}{\text{M1-day}}$	$200\ \dfrac{\$}{\text{M1-day}}$
Mine 2	$2\ \dfrac{\text{tons-HG}}{\text{M2-day}}$	$2\ \dfrac{\text{tons-MG}}{\text{M2-day}}$	$12\ \dfrac{\text{tons-LG}}{\text{M2-day}}$	$160\ \dfrac{\$}{\text{M2-day}}$
Requirements	$12\ \dfrac{\text{tons-HG}}{\text{week}}$	$8\ \dfrac{\text{tons-MG}}{\text{week}}$	$24\ \dfrac{\text{tons-LG}}{\text{week}}$	

Figure 9

	High-grade Ore	Medium-grade Ore	Low-grade Ore	Cost	
Mine 1	6	2	4	200	(per day)
Mine 2	2	2	12	160	(per day)
Requirements	12	8	24		(per week)
	(tons)	(tons)	(tons)	($)	

Figure 10

Let $v = (v_1, v_2)$ be the two-component row vector whose component v_1 gives the number of days per week that mine 1 operates and v_2 gives the number of days per week that mine 2 operates. If we define the quantities

$$A = \begin{pmatrix} 6 & 2 & 4 \\ 2 & 2 & 12 \end{pmatrix}, \qquad c = (12, 8, 24), \quad \text{and} \quad b = \begin{pmatrix} 200 \\ 160 \end{pmatrix},$$

which are immediately evident from the data box of Figure 9, we can state the problem above as a minimum problem.

Minimum problem: Determine the vector v so that the weekly operating cost, given by the quantity vb, is a minimum subject to the inequality restraints $vA \geq c$ and $v \geq 0$. The inequality restraints insure that the weekly output requirements are met and the limits on the components of v are not exceeded.

This is a minimum problem of the type discussed in detail in the preceding section. In Figure 11 we have graphed the convex polyhedral set C defined by the inequalities $vA \geq c$.

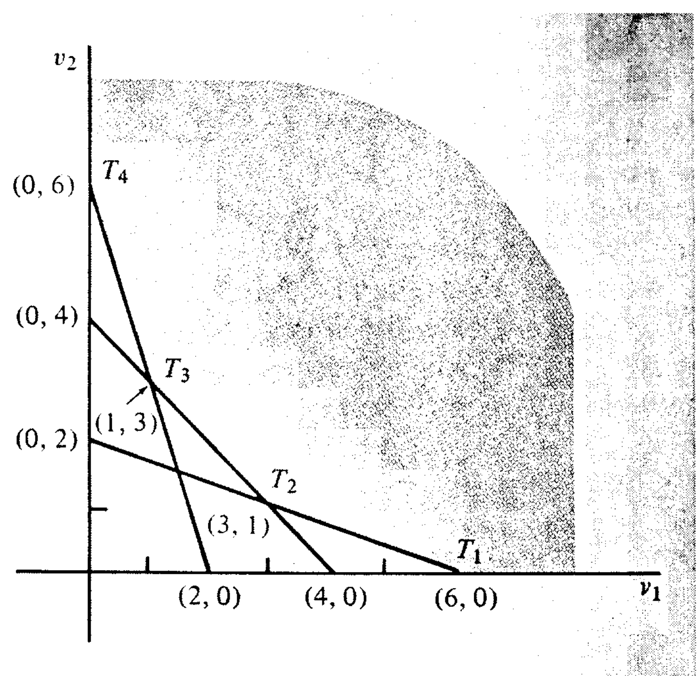


Figure 11

The extreme points of the convex set C are

$$T_1 = (6, 0), \qquad T_2 = (3, 1), \qquad T_3 = (1, 3), \qquad T_4 = (0, 6).$$

Testing the function $vb = 200v_1 + 160v_2$ at each of these extreme points, we see that it takes on the values 1200, 760, 680, and 960, respectively. We see that the minimum operating cost is \$680 per week and it is achieved at T_3—that is, by operating the first mine one day a week and the second mine three days a week.

Observe that if the mines are operated as indicated, then the combined weekly production will be 12 tons of high-grade ore, 8 tons of medium-grade ore, and 40 tons of low-grade ore. In other words, for this solution low-grade ore, is overproduced. If the company has no other demand for the low-grade ore, then it must discard 16 tons of it per week in this minimum-cost solution of its production problem. We shall discuss this point further in Section 6.

EXAMPLE 3 As a variant of Example 2, assume that the cost vector is

$$b = \begin{pmatrix} 160 \\ 200 \end{pmatrix};$$

in other words, the first mine now has a lower daily cost than the second. By the same procedure as above we find that the minimum cost level is again \$680 and is achieved by operating the first mine three days a week and the second mine one day a week. In this solution 20 tons of high-grade ore, instead of the required 12 tons, are produced, while the requirements of medium- and low-grade ores are exactly met. Thus 8 tons of high-grade ore must be discarded per week.

EXAMPLE 4 As another variant of Example 2, assume that the cost vector is

$$b = \begin{pmatrix} 200 \\ 200 \end{pmatrix};$$

in other words, both mines have the same production costs. Evaluating the cost function vb at the extreme points of the convex set, we find costs of \$1200 on two of the extreme points (T_1 and T_4) and costs of \$800 on the other two extreme points (T_2 and T_3). Thus the minimum cost is attained by operating either one of the mines three days a week and the other mine one day a week. But there are other solutions, since if the minimum is taken on at two distinct extreme points it is also taken on at each of the points on the line segment between. Thus any vector v where $1 \leq v_1 \leq 3$, $1 \leq v_2 \leq 3$, and $v_1 + v_2 = 4$ also gives a minimum-cost solution. For example, each mine could operate two days a week.

It can be shown (see Exercise 2) that for any solution v with $1 < v_1 < 3$, $1 < v_2 < 3$, and $v_1 + v_2 = 4$, both high-grade and low-grade ore are overproduced.

EXERCISES

1. In Example 1, assume that profits are \$200 per truck and \$300 per automobile. What should the factory now produce for maximum profit?
2. In Example 4, show that both high- and low-grade ore are overproduced for solution vectors v with $1 < v_1 < 3$, $1 < v_2 < 3$, and $v_1 + v_2 = 4$.
3. A manufacturer has two machines, M_1 and M_2, which he uses to

manufacture two products, P_1 and P_2. To produce one unit of P_1, three hours of time on M_1 and six hours on M_2 are needed. And to produce one unit of P_2 takes six hours on M_1 and five hours on M_2. Each machine can run a maximum of 2100 hours per year. If the manufacturer sells product P_1 for a net profit of \$40 and P_2 for a net profit of \$50 each, what production mix shall he produce to maximize his total profit?

(a) Set up the data box for the problem, marking the dimensions of all numbers.

(b) Find A, b, and c.

(c) Draw the set of possible production vectors and find the optimum profit point. [*Ans.* $x^0 = \begin{pmatrix}100\\300\end{pmatrix}$ with yearly profit of \$19,000.]

4. Two breakfast cereals, Krix and Kranch, supply varying amounts of vitamin B and iron; these are listed together with one-third of the daily minimum requirements (MDR) in the table below:

Cereal	Vitamin B	Iron
Krix	.15 mg/oz	1.67 mg/oz
Kranch	.10 mg/oz	3.33 mg/oz
$\frac{1}{3}$ MDR	.12 mg/day	2.0 mg/day

Krix costs 8 cents an ounce and Kranch 10 cents an ounce. How can we satisfy $\frac{1}{3}$ MDR at minimum cost?

(a) Let v_1 be the amount of Krix eaten and v_2 the amount of Kranch eaten. Write a minimizing linear programming problem for the above. Set up the data box and find A, b, and c.

(b) Draw the convex set of possible amounts eaten defined by the inequalities in (a).

(c) What is the lowest-cost feasible diet?

[*Ans.* $v^0 = (.6, .3)$ with cost 7.8 cents.]

5. A farmer owns a 200-acre farm and can plant any combination of two crops I and II. Crop I requires 1 man-day of labor and \$10 of capital for each acre planted, while crop II requires 4 man-days of labor and \$20 of capital for each acre planted. Crop I produces \$40 of net revenue per acre and crop II \$60. The farmer has \$2200 of capital and 320 man-days of labor available for the year. What is the optimal planting strategy? [*Ans.* $x^0 = \begin{pmatrix}180\\20\end{pmatrix}$ with \$8400 revenue.]

6. In Exercise 5 assume that the revenue from crop II is \$90 per acre.

(a) Find the new maximum-revenue scheme, and show that now the best thing for the farmer to do is to leave 30 acres unplanted.

(b) Explain why the farmer should leave part of his land fallow in this case.

7. Suppose that a pound of meat contains 1 unit of carbohydrates, 3 units of vitamins, and 12 units of proteins and costs \$1. Suppose also that one pound of cabbage contains 3, 4, and 1 units of these items, respectively, and costs 25 cents per pound. If these are the only foods available and the minimum daily requirements are 8 units of carbohydrates, 19 units of vitamins, and 7 units of protein, what is the minimum-cost diet? [*Ans.* $v^0 = (.2, 4.6)$ with cost \$1.35.]
8. Suppose that the minimum-cost diet found in Exercise 7 is unpalatable. In order to increase its palatability, add a constraint requiring that at least a half pound of meat be eaten, and resolve the problem. How much is the cost of the minimum-cost diet increased owing to this palatability constraint? [*Ans.* \$.24.]
9. In Exercise 8 suppose that we add a different kind of palatability constraint—namely, that at most two pounds of cabbage be eaten. Now how much is the cost of the minimum-cost diet increased? [*Ans.* \$2.82.]
10. A manufacturer produces two types of bearings, A and B, utilizing three types of machines; lathes, grinders, and drill presses. The machinery requirements for one unit of each product, in hours, are expressed in the following table:

	Machine		
Bearing	Lathe	Grinder	Drill Press
A	.01	.03	.03
B	.02	.01	.015
Weekly machine capacity (hr)	400	450	480

He makes a Profit of 10 cents per type A bearing and 15 cents per type B bearing. What should his weekly production of each bearing be in order to maximize his profits?

[*Ans.* $x = \begin{pmatrix} 8000 \\ 16{,}000 \end{pmatrix}$ with weekly profits of \$3200.]

4 THE DUAL PROBLEM

As the examples of the preceding sections have shown, some linear programming problems are maximizing and some are minimizing. Thus we might be interested in maximizing profits, production, or market share—or we might want to minimize costs, completion times, or raw-material usage. We shall show that to each maximizing problem there is a well-defined minimizing problem that uses the same data and whose solution has important mathematical implications concerning the original maximizing problem. Similarly, to each minimizing problem there is a well-defined maximizing problem that uses the same data and is similarly related.

First, we recall that every linear programming problem can be put into one of the two following forms:

$$\text{(1)} \qquad \left.\begin{array}{rr} \text{Maximize} & cx \\ \text{subject to} & Ax \leq b \\ & x \geq 0 \end{array}\right\} \text{the MAXIMUM problem.}$$

or

$$\text{(2)} \qquad \left.\begin{array}{rr} \text{Minimize} & vb \\ \text{subject to} & vA \geq c \\ & v \geq 0 \end{array}\right\} \text{the MINIMUM problem.}$$

If the components of A, b, c are the same, then the two problems (1) and (2) are called *dual linear programming problems*. Every linear programming problem, whether of the maximum or minimum type, has a dual that can be formally stated as above. The dual of a given problem frequently has important economic meaning and always has mathematical significance—see the discussion in Section 6.

To set up a *maximum problem* proceed as follows: Let the variables to be determined be $x_1, \ldots, x_n$; set up the data box as in Figure 12, with the x-variables appearing as labels on the top of the box. It then follows that, taking A, b, and c from the data box, the maximum problem is in form (1) above.

x_1	x_2	$\cdots$	x_n	
a_{11}	a_{12}	$\cdots$	a_{1n}	b_1
a_{21}	a_{22}	$\cdots$	a_{2n}	b_2
$\cdot$	$\cdot$	$\cdots$	$\cdot$	$\cdot$
a_{m1}	a_{m2}	$\cdots$	a_{mn}	b_m
c_1	c_2	$\cdots$	c_n	

Figure 12

To set up a *minimum problem* proceed as follows: Let the variables to be determined by $v_1, \ldots, v_m$; set up the data box as in Figure 13 with the v-variables appearing as labels to the left of the box. It then follows that, taking A, b, and c from the data box, the minimum problem is in form (2) above.

v_1	a_{11}	a_{12}	$\cdots$	a_{1n}	b_1
v_2	a_{21}	a_{22}	$\cdots$	a_{2n}	b_2
$\cdot$	$\cdot$	$\cdot$	$\cdots$	$\cdot$	$\cdot$
v_m	a_{m1}	a_{m2}	$\cdots$	a_{mn}	b_m
	c_1	c_2	$\cdots$	c_n	

Figure 13

We now make two important observations. First, the dual problem to a maximum problem with data box as in Figure 12 can be obtained by merely labeling the rows $v_1, \ldots, v_m$; and the dual problem to the minimum problem whose data box is in Figure 13 can be obtained by labeling the columns $x_1, \ldots, x_n$. Second, the dimensions of the dual variables in either case can be obtained by dividing the dimensions of the b's or c's by the corresponding a's, as the following examples will make clear. We shall see that the interpretations of the dual variables are easy, once their physical dimensions are determined.

The reader may wonder why we introduce the dual problem instead of concentrating on the original problem alone. The reason is that the simplex method to be discussed later automatically produces the optimum solution to both problems simultaneously. Also, the solution to the dual problem often has important managerial and economic interpretations.

Before we can describe how the simplex method works, we must make a change in the formulation of the dual programs. What we shall do is to add *slack* variables to the inequalities stated in expressions (1) and (2) of this section in such a way as to make them into equations. To see how this is done, consider as an example the system of inequalities

$$-u + 2v \leq 5, \qquad \text{where } u \geq 0 \text{ and } v \geq 0.$$

We now add a new slack variable w and obtain a new system of expressions:

$$-u + 2v + w = 5, \qquad \text{where } u \geq 0,\ v \geq 0,\ w \geq 0.$$

Thus we obtain the equation

$$-u + 2v + w = 5$$

in nonnegative variables. Notice that the new system of expressions is equivalent to the old system, since any solution of the new system that has $w = 0$ represents a case for which $-u + 2v = 5$, and a solution of the new system for which $w > 0$ represents a case for which $-u + 2v < 5$. Moreover, we can write any solution of the old system as a solution of the new system by properly choosing a nonnegative value of w. Thus the truth sets of the two systems are identical.

Now we want to reformulate the constraints of problems (1) and (2). Let y be an m-component vector of *slack variables* y_i, and let f be a number; then (1) is equivalent to

$$\text{(3)} \qquad \begin{aligned} &\text{Maximize} && cx = f \\ &\text{subject to} && Ax + y = b, \\ & && x, y \geq 0. \end{aligned}$$

To see this, rewrite the constraint of (3) as follows:

$$\text{(4)} \qquad Ax - b = -y;$$

then $y \geq 0$ is equivalent to $-y \leq 0$, and the latter is, from (4), the same as $Ax \leq b$. The number $f = cx$ measures the current value of the objective function of the maximum problem.

Similarly, let u be an n-component row vector of *slack* variables u_j, and let g be a number; then (2) is equivalent to

$$\begin{array}{lrl} & \text{Minimize} & vb = g \\ (5) & \text{subject to} & vA - u = c, \\ & & u, v \geq 0. \end{array}$$

To see the equivalence rewrite the equality constraint of (5) as

$$(6) \qquad vA - c = u;$$

then it is obvious that $u \geq 0$ and $vA \geq c$ are the same. The number $g = vb$ measures the current value of the objective function of the minimizing problem.

Next we show that the pair of dual problems in (3) and (5) can both be represented in the same tableau, and that the tableau can be obtained by extending either of the data boxes in Figure 12 or 13. Consider the (Tucker) tableau, which we shall later call the *initial simplex tableau*, in Figure 14.

$$\begin{array}{c|cccc|c|c} & x_1 & x_2 & \cdots & x_n & -1 & \\ \hline v_1 & a_{11} & a_{12} & \cdots & a_{1n} & b_1 & = -y_1 \\ v_2 & a_{21} & a_{22} & \cdots & a_{2n} & b_2 & = -y_2 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ v_m & a_{m1} & a_{m2} & \cdots & a_{mn} & b_m & = -y_m \\ \hline -1 & c_1 & c_2 & \cdots & c_n & 0 & = f \\ \hline & = u_1 & = u_2 & \cdots & = u_n & = g & \end{array}$$

Figure 14

Notice that Figure 14 can be obtained from Figure 12 by adding the 0 entry in the lower right-hand corner, putting variables $v_1, \ldots, v_m$ and -1 along the left margin, putting -1 above the $(n + 1)$st column, marking the right-hand side with $= -y_1, \ldots, = -y_m$, and $= f$, and marking the bottom of the matrix with $= u_1, \ldots, = u_n$, and $= g$. Figure 14 can be obtained in a similar manner from Figure 13. The reason for this labeling is as follows: if we drop the x's and -1 down to the first row of the matrix, multiply by the coefficients there, and set equal to the label on the right, we have

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n - b_1 = -y_1,$$

which is just exactly the first equation of (4). Dropping the labels at the top down to the other rows will give the other equations of (4). Finally, dropping the labels down to the last row gives

$$c_1x_1 + c_2x_2 + \ldots + c_nx_n = f,$$

which is just the definition of f.

In a similar manner, if we move the labels on the left of Figure 14 into each column of the tableau, multiply, and set equal to the label at the bottom, we have the various equations of (6) together with the definition of g.

EXAMPLE 1 The data box for the automobile/truck example of the last section is shown in Figures 6 and 7; hence its initial simplex tableau is as given in Figure 15.

	x_1	x_2	-1	
v_1	5	2	180	$= -y_1$
v_2	3	3	135	$= -y_2$
-1	300	200	0	$= f$
	$= u_1$	$= u_2$	$= g$	

Figure 15

The primal equations for this problem corresponding to (4) are

$$\begin{aligned} 5x_1 + \quad 2x_2 - 180 &= -y_1, \\ 3x_1 + \quad 3x_2 - 135 &= -y_2, \\ 300x_1 + 200x_2 \qquad &= f. \end{aligned}$$

The dual equations for this problem corresponding to (6) are

$$\begin{aligned} 5v_1 + \quad 3v_2 - 300 &= u_1, \\ 2v_1 + \quad 3v_2 - 200 &= u_2, \\ 180v_1 + 135v_2 \qquad &= g. \end{aligned}$$

These are obtained in the manner described above.

EXAMPLE 2 The data box for the mining example of the last section is shown in Figures 9 and 10; hence its initial simplex tableau is as given in Figure 16.

	x_1	x_2	x_3	-1	
v_1	6	2	4	200	$= -y_1$
v_2	2	2	12	160	$= -y_2$
-1	12	8	24	0	$= f$
	$= u_1$	$= u_2$	$= u_3$	$= g$	

Figure 16

The primal equations for this problem corresponding to (6) are

$$\begin{aligned} 6v_1 + \quad 2v_2 - 12 &= u_1, \\ 2v_1 + \quad 2v_2 - \ \ 8 &= u_2, \\ 4v_1 + \ 12v_2 - 24 &= u_3, \\ 200v_1 + 160v_2 \qquad &= g, \end{aligned}$$

and the dual equations corresponding to (4) are

$$\begin{aligned} 6x_1 + 2x_2 + 4x_3 - 200 &= -y_1, \\ 2x_1 + 2x_2 + 12x_3 - 160 &= -y_2, \\ 12x_1 + 8x_2 + 24x_3 &= f. \end{aligned}$$

The reader should set up in an analogous way the initial simplex tableaus for Examples 3 and 4 of Section 3.

We next show that from equations (4) and (6) we can immediately derive *Tucker's duality equation:*

$$g - f = vy + ux. \tag{7}$$

This follows easily since

$$g - f = vb - cx = v(Ax + y) - (vA - u)x = vy + ux,$$

where we used the substitutions $b = Ax + y$ from (4) and $c = vA - u$ from (6).

Nonnegative vectors x, y, u, and v that satisfy (4) and (6) will be called *feasible vectors* for the equality form of the linear programming problem. Note that the duality relation (7) is true for *all* solutions x, y, u, and v satisfying (4) and (6) whether nonnegative or not. However, the following theorem shows that a pair of feasible vectors for one of the problems implies a bound on the objective function of the other problem.

Theorem (a) Let x^0, y^0, and f^0 be *optimal* solutions to maximizing problem (3), and let u, v, and g be *feasible* solutions to the dual minimizing problem (5); then $cx^0 = f^0 \leq g = vb$; in other words, for any feasible vector v, the value $g = vb$ is an *upper bound* to the maximum value $f^0 = cx^0$ of the maximizing problem (3).

(b) Let u^0, v^0, and g^0 be *optimal* solutions to the minimizing problem (5), and let x, y, and f be *feasible* solutions to the dual maximizing problem (3); then $v^0b = g^0 \geq f = cx$; in other words, for any feasible vector x, the value $f = cx$ is a *lower bound* to the minimum value $g^0 = v^0b$ of the minimizing problem (5).

Proof (a) If u, v, x^0, and y^0 are all nonnegative vectors, then it follows that $vy^0 \geq 0$ and $ux^0 \geq 0$, so that, from (7), we have

$$g - f^0 = vy^0 + ux^0 \geq 0,$$

or, in other words, $g \geq f^0$, as asserted.

The proof of (b) is similar.

We illustrate the theorem by returning to the previous examples.

EXAMPLE 1 (continued) If we consider the automobile/truck example whose initial tableau is given in Figure 15, we can easily check that the following quantities solve the primal problem: $x_1 = 10, x_2 = 10, y_1 = 110, y_2 = 75$. These were obtained by selecting arbitrary but not too large values for x_1 and x_2 and then solving for y_1 and y_2. From this feasible solution we calculate $cx = 300 \cdot 10 + 200 \cdot 10 = 3000 + 2000 = 5000$; hence we know that $5000 \leq g^0 = v^0 b$; that is, we have found a lower bound to the optimum value g^0 of the dual minimizing problem.

Similarly, we can select v_1 and v_2 to be fairly large, but otherwise arbitrary, and solve for u_1 and u_2. For instance, $v_1 = 50, v_2 = 40, u_1 = 70$, and $u_2 = 20$ are a feasible choice for these quantities. From them we know that $f^0 = cx^0$ is definitely not greater than $vb = 180 \cdot 50 + 135 \cdot 40 = 9000 + 5400 = 14{,}400$.

Since we know that the optimum value is $f^0 = 12{,}000$, and we will later show that $f^0 = g^0$, we see that, in fact, the lower and upper bounds are correct in this instance. The reader should try several other feasible solutions for this example.

EXAMPLE 2 (continued) Let us check the theorem for the mining example shown in Figure 16. Suppose we choose $x_1 = 20, x_2 = 20, x_3 = 5$, so that $y_1 = 20$ and $y_2 = 20$. We thus obtain the lower bound on g^0 as $cx = 12 \cdot 20 + 8 \cdot 20 + 24 \cdot 5 = 240 + 160 + 120 = 520$.

Similarly, we can choose $v_1 = 2$, $v_2 = 2$, and correspondingly $u_1 = 4$, $u_2 = 0$, and $u_3 = 8$, so an upper bound for f^0 is $vb = 200 \cdot 2 + 160 \cdot 2 = 720$.

Since the true value is 680, we see that the upper and lower bounds again are correct.

EXERCISES

1. Illustrate the theorem of this section by finding other feasible solutions to the primal and dual problems for the automobile/truck example, and show that the upper and lower bounds so obtained are correct.
2. Repeat Exercise 1 for the mining example.
3. For Example 3 of Section 3:
   (a) Set up and label the initial tableau.
   (b) Write the primal and dual equations.
   (c) Find feasible solutions to the primal equations and determine a bound to the dual problem.
   (d) Find feasible solutions to the dual problem and derive a bound on the primal problem.
4. Repeat Exercise 3 for Exercise 4 of Section 3.
5. Repeat Exercise 3 for Exercise 5 or Section 3.
6. Repeat Exercise 3 for Exercise 7 of Section 3.
7. Repeat Exercise 3 for Exercise 10 of Section 3.

*8. Let x^0 and v^0 be nonnegative vectors such that $f^0 = cx^0 = v^0 b = g^0$, $Ax^0 \leq b$, and $v^0 A \geq c$.

(a) Show that if x is any other feasible vector, then

$$cx \leq v^0 Ax \leq v^0 b = cx^0,$$

so that x^0 solves the maximum problem.

(b) Similarly, show that v^0 solves the minimum problem.

(c) Show that $cx^0 = v^0 b = v^0 Ax^0$.

9. Use (7) to show that if x, y, u, and v are vectors related as in (4) and (6), then $ux \geq 0$ and $vy \geq 0$ imply $g \geq f$. (Note that this is true whether or not x, y, u, and v are nonnegative.)

If x, y, u, and v are vectors related as in (4) and (6), then they are said to have the *complementary slackness property* if and only if

$$ux = 0 \quad \text{and} \quad vy = 0.$$

The remaining exercises refer to this property.

10. Use (7) to show that if x, y, u, and v satisfy the complementary slackness property, then $g = f$. Is the converse true?

***11.** If x, y, u, and v are nonnegative vectors, show that $g = f$ if and only if they have the complementary slackness property.

***12.** Use Exercises 8, 10, and 11 to show that nonnegative vectors related as in (4) and (6) are optimal if and only if they satisfy the complementary slackness property.

5 THE SIMPLEX METHOD

In Section 3 we solved simple linear programming problems having two variables by sketching convex sets in the plane. To solve such problems in more than two variables by the same method would require visualizing convex sets in more than two dimensions, which is extremely difficult. But fortunately there is an algorithm, called the *simplex algorithm,* that permits us to solve such large-scale linear programming problems without such visualizations. The reader will recall that in Chapter 4 we developed an algorithm for solving simultaneous linear equations that was algebraic (not geometric) in nature and avoided similar visualization problems.

For simplicity we shall make the following two assumptions in the present and next sections:

I. The Nonnegativity Assumption We shall assume $b \geq 0$; that is, every component of b is nonnegative.

II. The Nondegeneracy Assumption The extreme points of the convex set of feasible vectors are each the intersection of *exactly* n bounding hyperplanes, where n is the number of components of the vectors involved.

In Section 7 we shall indicate how these two assumptions can be dropped. We emphasize, however, that for linear programming problems derived from actual applications both assumptions will be satisfied, or else the problem can be reformulated so that they are. Moreover, when codes are written for computers to solve linear programming problems, precautions are taken to insure that these assumptions hold.

We now proceed to describe the simplex method. In the next section we shall discuss reasons why the simplex method works.

After the data box has been set up for either a maximizing or minimizing problem, the simplex method begins with the initial simplex tableau (the Tucker tableau) of Figure 14. Note that it was derived from the data box as described in the previous section. The simplex algorithm will change the initial tableau into a second one, that into a third, and so on, until finally a tableau is obtained that displays the optimum answers to both the primal and dual problems. A typical tableau in this computational process is shown in Figure 17. Note that the variables have been identified as being of two

$$
\begin{array}{r|cccc|c|ll}
 & \multicolumn{4}{c|}{\overbrace{\qquad\qquad\qquad\qquad}^{\text{Nonbasic } x\text{- and } y\text{-variables}}} & -1 & & \\
\hline
\text{Nonbasic} & t_{11} & t_{12} & \cdots & t_{1n} & t_{1,n+1} & = - & \text{Basic} \\
u\text{- and } v\text{-} & t_{21} & t_{22} & \cdots & t_{2n} & t_{2,n+1} & = - & x\text{- and } y\text{-} \\
\text{variables} & \cdot & \cdot & \cdots & \cdot & \cdot & & \text{variables} \\
 & t_{m1} & t_{m2} & \cdots & t_{mn} & t_{m,n+1} & = - & \\
\hline
-1 & t_{m+1,1} & t_{m+1,2} & \cdots & t_{m+1,n} & t_{m+1,n+1} & = f & \\
 & \multicolumn{4}{c|}{\text{Indicators}} & & & \\
\hline
 & = & = & & = & = g & & \\
 & \multicolumn{4}{c}{\underbrace{\qquad\qquad\qquad\qquad}_{\text{Basic } u\text{- and } v\text{-variables}}} & & &
\end{array}
$$

Figure 17

kinds: *basic* and *nonbasic*. The basic variables appear on the bottom and right-hand sides of the tableau and the nonbasic variables on the left and top. As we shall see, in any tableau, if we set the nonbasic variables equal to zero, then the corresponding values of the basic variables can be read from the last row and last column of the tableau. The other important thing to note is that the entries of the first n columns of the last row are called *indicators*.

The flow chart in Figure 18 describes how the simplex method works. Look at box 1 in the upper left-hand corner. We see that for the automobile/truck and mining examples of the previous section we have already carried out the directives there: the problems are set up and the initial tableaus formed. We now discuss in detail the rest of the computation for these two examples.

EXAMPLE 1 The initial tableau for the automobile/truck example appears in Figure 15. To solve this problem using the simplex method we go next to box 2 of the flow chart in Figure 18. We note that in the initial simplex tableau of

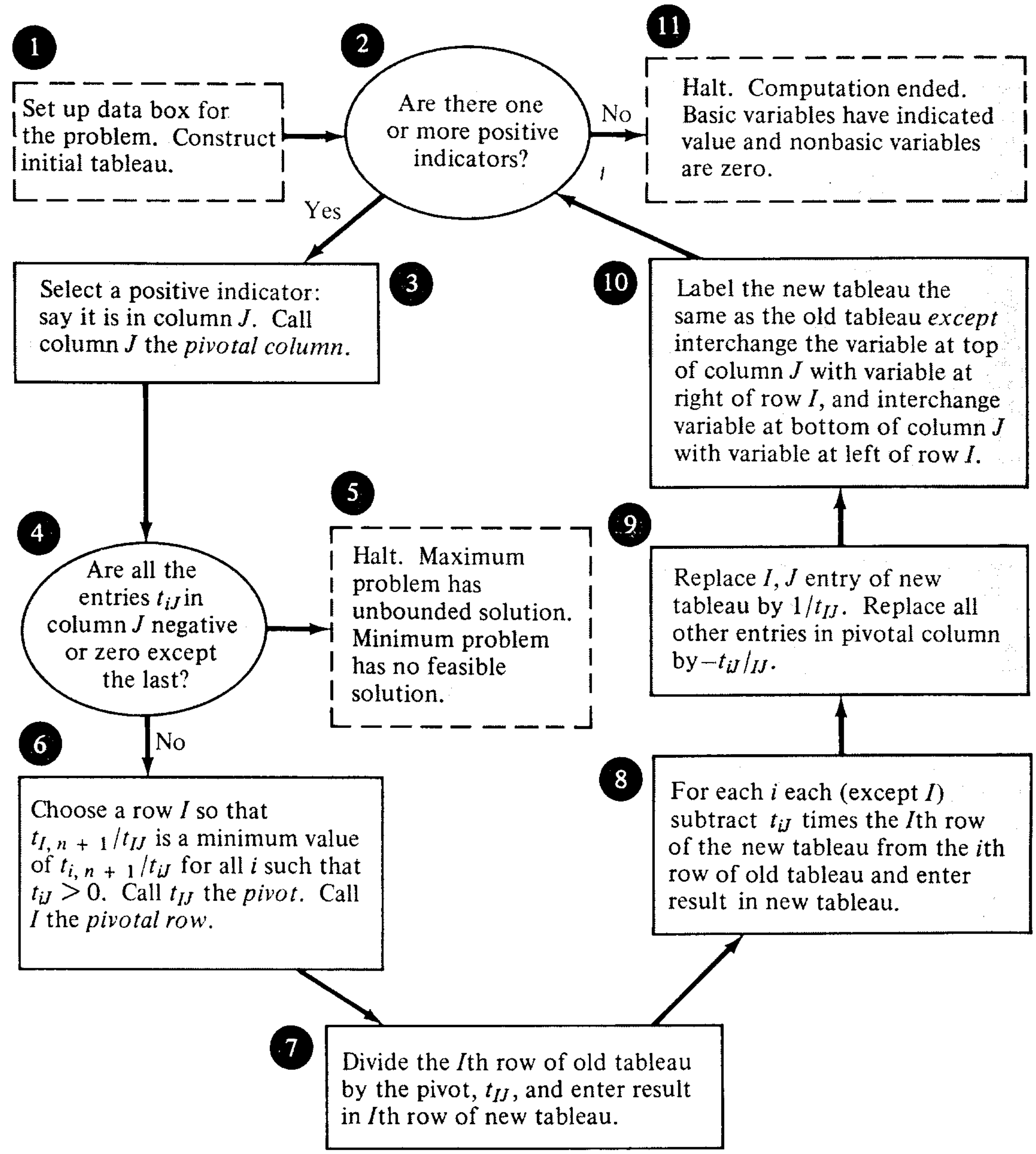


Figure 18 The simplex algorithm for problems with nonnegative righthand sides

Figure 15 there are positive indicators, so the answer to the question in box 2 is "yes." Hence we proceed to box 3, which says, "Select a positive indicator." Suppose we select 300, which makes column 1 the pivotal column and $J = 1$. We now go to box 4 and observe that there are positive entries in column 1, so that the answer to the question there is "no," and we go on to box 6. We must now find the pivotal row. For this we examine the ratios $t_{i,n+1}/t_{i1}$ for $i = 1$ and 2. These ratios are $180/5 = 36$ and $135/3 = 45$. Since the smaller ratio occurs in the first row, we see that the 5 entry in the first column of Figure 15 is the pivot and $I = 1$, so that the first row is pivotal. The pivot is circled in Figure 15.

Next we carry out the directives in boxes 7 and 8 of Figure 18, which construct the rows of the new tableau. In box 7 we find we must divide through the pivotal row of the old tableau by the pivot and insert it in the new tableau (Figure 19). Then we multiply this new row by 3 and subtract

1	$\frac{2}{5}$	36
0	$\frac{9}{5}$	27
0	80	$-10{,}800$

Figure 19

it from the second row of the old tableau to form the second row of the new tableau. In vector form, this computation is

$$-3(1 \quad \tfrac{2}{5} \quad 36) + (3 \quad 3 \quad 135) = (0 \quad \tfrac{9}{5} \quad 27).$$

Similarly, we multiply the new row by 300 and subtract from the third row of the old tableau to form the third row of the new tableau as shown. To complete the new tableau we must replace the pivotal column as described in box 9 of Figure 18; the result is given in Figure 20. Also we must

	y_1	x_2	-1	
u_1	$\frac{1}{5}$	$\frac{2}{5}$	36	$= -x_1$
v_2	$-\frac{3}{5}$	$\frac{9}{5}$	27	$= -y_2$
-1	-60	80	$-10{,}800$	$= f$
	Indicators			
	$= v_1$	$= u_2$	$= g$	

Figure 20

interchange the labels of the variables at both ends of the pivot row with the variables at both ends of the pivot column as described in box 10 of Figure 18. The completed new second tableau appears in Figure 20.

We now find ourselves back at box 2 of the flow chart of Figure 18. Since the 80 in the second column, last row of Figure 20 is positive, the answer to the question in box 2 is "yes," so we go to box 3. Clearly we must choose $J = 2$. The answer to question in box 4 is "no," so we go on to box 6 to choose the pivot. The two ratios to be considered are $36/\frac{2}{5} = 90$ and $27/\frac{9}{5} = 15$, so that the second row is pivotal and $\frac{9}{5}$ (circled in Figure 20) is the new pivot. Carrying out the instructions in boxes 7 and 8 of the flow diagram gives the tableau in Figure 21, and finishing up with boxes 9 and 10 gives the completed third tableau (Figure 22).

$\frac{1}{3}$	0	30
$-\frac{1}{3}$	1	15
$-\frac{100}{3}$	0	$-12{,}000$

Figure 21

$$
\begin{array}{c|cc|c|l}
 & y_1 & y_2 & -1 & \\
\hline
u_1 & \frac{1}{3} & -\frac{2}{9} & 30 & = -x_1 \\
u_2 & -\frac{1}{3} & \frac{5}{9} & 15 & = -x_2 \\
\hline
-1 & \multicolumn{2}{c|}{\underbrace{-\frac{100}{3} \qquad -\frac{400}{9}}_{\text{Indicators}}} & -12{,}000 & = f \\
\hline
 & = v_1 & = v_2 & = g &
\end{array}
$$

Figure 22

We again find ourselves in box 2 of the flow diagram. But this time we find no positive indicators for the tableau of Figure 22; hence the answer to the question there is "no" and we go to box 11, which says that the computation is ended. The answers to both the primal and dual problems are displayed in the final tableau. To see what they are, we first set the nonbasic variables equal to zero as instructed in box 11 of the flow diagram. Hence we have $u_1 = u_2 = y_1 = y_2 = 0$, since the nonbasic variables appear on the left and top of the final tableau. Knowing that $y_i = 0$ for $i = 1, 2$, we drop the variables at the top of the final tableau down to the first row and multiply, obtaining $-30 = -x_1$ or simply $x_1 = 30$. Dropping these down one row further gives $x_2 = 15$. And dropping them down to the last row gives $f = 12{,}000$, which is the final value of the objective function. Thus the optimal solution vectors to the maximizing problem are:

$$x^0 = \begin{pmatrix} 30 \\ 15 \end{pmatrix}, \qquad y^0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \text{and} \quad f^0 = 12{,}000.$$

Note that this is the same solution that we found in the previous section.

We can also find the optimal solution to the dual problem. (The interpretation of this solution will be given in the next section.) Knowing $u_j = 0$ for $j = 1, 2$, we move the variables on the left of Figure 22 into the first column, multiply, and obtain $v_1 = \frac{100}{3}$. Moving them to the second column gives $v_2 = \frac{400}{9}$, and moving them to the third column gives $g = 12{,}000$, the value of the objective function of the minimizing problem. Hence the optimal solution vectors to the minimizing problem are:

$$v^0 = (\tfrac{100}{3}, \tfrac{400}{3}), \qquad u^0 = (0, 0), \quad \text{and} \quad g^0 = 12{,}000.$$

The reader should substitute x^0 and y^0 into the primal, and v^0 and u^0 into the dual equations written down previously and show that they are satisfied. Note also that $f^0 = v^0 b = cx^0 = g^0$ at an optimum solution. This is always true, and will be discussed further in the next section.

EXAMPLE 2 Let us solve the mining example using the simplex method. The initial tableau is in Figure 16. The first indicator 12 was selected so that the first column is pivotal. The pivot is 6, which is circled in the figure, and was chosen because the two ratios involved are $\frac{100}{3}$, which is smaller than $\frac{160}{2} = 80$, hence the first row is pivotal and the pivot is 6. Carrying out steps 7 through 10 of the flow diagram (Figure 18), we construct the second

$$
\begin{array}{c|ccc|c|l}
 & y_1 & x_2 & x_3 & -1 & \\
\hline
u_1 & \frac{1}{6} & \frac{1}{3} & \frac{2}{3} & \frac{100}{3} & = -x_1 \\
v_2 & -\frac{1}{3} & \frac{4}{3} & \frac{32}{3} & \frac{280}{3} & = -y_2 \\
\hline
-1 & -2 & 4 & 16 & -400 & = f \\
 & \multicolumn{3}{c|}{\text{Indicators}} & & \\
\hline
 & = v_1 & = u_2 & = u_3 & = g &
\end{array}
$$

Figure 23

tableau in Figure 23. There are two positive indicators, and we choose the first one, 4, so that the second column is pivotal. The new pivot is $\frac{4}{3}$, which is circled in the second (pivotal) row. Carrying out the rest of the steps of the flow diagram, we obtain the third tableau (Figure 24). All indicators

$$
\begin{array}{c|ccc|c|l}
 & y_1 & y_2 & x_3 & -1 & \\
\hline
u_1 & \frac{1}{4} & -\frac{1}{4} & -2 & 10 & = -x_1 \\
u_2 & -\frac{1}{4} & \frac{3}{4} & 8 & 70 & = -x_2 \\
\hline
-1 & -1 & -3 & -16 & -680 & = f \\
 & \multicolumn{3}{c|}{\text{Indicators}} & & \\
\hline
 & = v_1 & = v_2 & = u_3 & = g &
\end{array}
$$

Figure 24

in this tableau are negative, so the computation is complete. We read off the optimum answers to the primal minimizing problem as

$$v^0 = (1, 3), \qquad u^0 = (0, 0, 16), \quad \text{and} \quad g^0 = 680,$$

and the final minimum operating cost for the mines is \$680 per week. These are the same answers as the graphical procedure gave.

The optimum answers to the dual maximizing problem can also be obtained as

$$x^0 = \begin{pmatrix} 10 \\ 70 \\ 0 \end{pmatrix}, \quad y^0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \text{and} \quad f^0 = 680.$$

Interpretations for these will be given in the next section.

EXAMPLE 3 Our next example illustrates the fact that a given variable may first be basic, become nonbasic, then become basic again, and so on, several times during the course of the simplex computation. Figures 25 through 28 give the necessary tableaus, and the pivots are circled there. There is another way of working this problem that requires only two tableaus. It starts with a pivot in the first column instead of the second (see Exercise 9). This illustrates the rule that it is frequently (but not invariably) better to start the simplex method with a column having the most positive indicator. Note that y_1 started out basic, became nonbasic, then became basic again. And

$$\begin{array}{c|cc|c|l} & x_1 & x_2 & -1 & \\ \hline v_1 & 2 & 1 & 3 & = -y_1 \\ v_2 & 3 & 1 & 4 & = -y_2 \\ \hline -1 & 17 & 5 & 0 & = f \\ \hline & = u_1 & = u_2 & = g & \end{array}$$

Figure 25

$$\begin{array}{c|cc|c|l} & x_1 & y_1 & -1 & \\ \hline u_2 & 2 & 1 & 3 & = -x_2 \\ v_2 & 1 & -1 & 1 & = -y_2 \\ \hline -1 & 7 & -5 & -15 & = f \\ \hline & = u_1 & = v_1 & = g & \end{array}$$

Figure 26

$$\begin{array}{c|cc|c|l} & y_2 & y_1 & -1 & \\ \hline u_2 & -2 & 3 & 1 & = -x_2 \\ u_1 & 1 & -1 & 1 & = -x_1 \\ \hline -1 & -7 & 2 & 22 & = f \\ \hline & = v_2 & = v_1 & = g & \end{array}$$

Figure 27

$$\begin{array}{c|cc|c|l} & y_2 & x_2 & -1 & \\ \hline v_1 & -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} & = -y_1 \\ u_1 & \frac{1}{3} & \frac{1}{3} & \frac{4}{3} & = -x_1 \\ \hline -1 & -\frac{17}{3} & -\frac{2}{3} & -22\frac{2}{3} & = f \\ \hline & = v_2 & = u_2 & = g & \end{array}$$

Figure 28

x_2 was initially nonbasic, became basic, and ended up nonbasic. The final optimal answers are:

$$v^0 = (0, \tfrac{17}{3}), \qquad u^0 = (0, \tfrac{2}{3}), \qquad g^0 = 22\tfrac{2}{3},$$

$$x^0 = \begin{pmatrix} \frac{4}{3} \\ 0 \end{pmatrix}, \qquad y^0 = \begin{pmatrix} \frac{1}{3} \\ 0 \end{pmatrix}, \qquad f^0 = 22\tfrac{2}{3}.$$

EXAMPLE 4 The reader has undoubtedly wondered about box 5 of the flow diagram in Figure 18, since we have not yet ended in it. Actually, if we are solving an applied problem that is correctly formulated so that it has a solution, we shall never end in it. Consider, however, the problem whose initial tableau is in Figure 29. Both the first two columns have positive indicators. If we choose the first one and pivot, we obtain the tableau of Figure 30. Now there is one positive indicator in the second column, so $J = 2$. But

$$
\begin{array}{c|cc|c|l}
 & x_1 & x_2 & -1 & \\
\hline
v_1 & -1 & 1 & 1 & = -y_1 \\
v_2 & 1 & -1 & 1 & = -y_2 \\
\hline
-1 & 1 & 1 & 0 & = f \\
\hline
 & = u_1 & = u_2 & = g &
\end{array}
$$

Figure 29

$$
\begin{array}{c|cc|c|l}
 & y_2 & x_2 & -1 & \\
\hline
v_1 & 1 & 0 & 2 & = -y_1 \\
u_1 & 1 & -1 & 1 & = -x_1 \\
\hline
-1 & -1 & 2 & -1 & = g \\
\hline
 & = v_2 & = u_2 & = g &
\end{array}
$$

Figure 30

the answer to the question in box 4 of Figure 18 is "yes," so we arrive at box 5, which says that the maximum problem has an unbounded solution and the minimum problem has no feasible solution.

To see this let us write the constraints for the maximum problem of Figure 29. They are

$$
\begin{aligned}
-x_1 + x_2 &\leq 1, \qquad & x_1 &\geq 0, \\
x_1 - x_2 &\leq 1, \qquad & x_2 &\geq 0.
\end{aligned}
$$

These inequalities are satisfied if x_1 and x_2 are equal and positive. Hence we can make the objective function $f = x_1 + x_2$ as large as we wish. Two constraints of the minimum problem of Figure 29 are

$$
\begin{aligned}
-v_1 + v_2 &\geq 1, \\
v_1 - v_2 &\geq 1
\end{aligned}
$$

If we add these, we obtain the contradiction $0 \geq 2$, and hence the minimum problem has no solution.

For practical purposes, however, we can ignore the no-solution possibility, since we will be dealing with well-formulated problems that have solutions.

EXERCISES

1. Use the simplex method to solve Example 3 of Section 3.
2. Use the simplex method to solve Example 4 of Section 3 even though the nondegeneracy hypothesis is not satisfied. Show that there are two ways to proceed, each one leading to a different solution of the minimum problem.
3. Use the simplex method to solve Exercise 3 of Section 3.
4. Use the simplex method to solve Exercise 4 of Section 3.
5. Use the simplex method to solve Exercise 5 of Section 3.
6. Use the simplex method to solve Exercise 6 of Section 3.

7. Use the simplex method to solve Exercise 7 of Section 3.
8. Use the simplex method to solve Exercise 8 of Section 3.
9. Solve the problem in Example 3 by choosing the first pivot in the first column. Show that the answer can be obtained in one step.
10. Use the simplex method to solve Exercise 10 of Section 3.
11. A nut packager has on hand 121 pounds of peanuts and 49 pounds of cashews. He can sell two kinds of mixtures of these nuts: a cheap mix that has 80 percent peanuts and 20 percent cashews, or a party mix that has 30 percent peanuts and 70 percent cashews. If he can sell the party mix at 80 cents a pound and the cheap mix at 50 cents a pound, how many pounds of each mix should he make in order to maximize the amount he can obtain?
[*Ans.* Let x_1 be the number of pounds of party mix and x_2 the number of pounds of the cheap mix. Then the data are

$$A = \begin{pmatrix} .3 & .8 \\ .7 & .2 \end{pmatrix}, \qquad b = \begin{pmatrix} 121 \\ 49 \end{pmatrix}, \quad \text{and} \quad c = (80, 50).$$

The packager should make 30 pounds of the party mix and 140 pounds of the cheap mix. His income is \$94.]
12. The operator of all oil refinery can buy light crude oil at \$6 per barrel and heavy crude at \$5 per barrel. The refining process produces the following quantities of gasoline, kerosene, and fuel oil from one barrel of each type of crude:

Type	*Gasoline*	*Kerosene*	*Fuel Oil*
Light crude	.5	.25	.2
Heavy crude	.4	.3	.25

Note that in each case 5 percent of the barrel of crude is lost in the form of gases (which have to be burned) and unusable sludge. During the summer months the operator has contracted to deliver 50,000 barrels of gasoline, 30,000 barrels of kerosene, and 10,000 barrels of fuel oil per month. How many barrels of each type of crude should he process in order to meet his production quotas at minimum possible cost?
13. During the winter months the refinery operator of Exercise 12 contracts to deliver 36,000 barrels of gasoline, 12,000 barrels of kerosene, and 18,000 barrels of fuel oil. What is his optimal winter production plan?
14. In Exercises 12 and 13 show that there is an excess production of at least one of the goods during each time of the year. Discuss practical ways in which this excess production can be used.
15. In the tableau of Figure 16 make the pivot be the 2 entry in the first column rather than the circled 6 entry shown. Show that this leads to a negative value of x_1, and hence explain the reasons in box 6 of Figure 18 for the special choice of the pivot.

6 DUALITY INTERPRETATIONS AND RESULTS

As we saw in the previous section, the simplex method is the same for both maximizing and minimizing problems. The only difference in setting up the two problems is the choice of row or column vectors for the various quantities involved. In either case we ended up with a data box containing a matrix A, a column vector b, and a row vector c. Using these data we stated both a maximizing and a minimizing problem—only one of which initially interested us. The other problem is called the *dual* linear programming problem. The dual of a maximizing problem is a minimizing problem, and vice versa. And the dual of the dual problem is, in either case, the original problem.

We saw that the simplex method solves both the original problem and its dual simultaneously. It is therefore of interest to see what interpretation, if any, can be given to the dual of a linear programming problem. We shall see that we can always give the dual problem mathematical and economic or managerial interpretations that are of considerable interest.

The first step in interpreting the solution to the dual problem is that of determining the dimensions of the variables involved. Recall that in Section 3 we set up for each linear programming problem a data box, and the numbers in the data box had dimensions. We now need to determine the dimensions of the variables of both the primal and dual problems. The following rule tells how to do this.

Rule for Determining Dimensions of Variables

(a) The dimension of x_j is the ratio of the dimension of b_i divided by the dimension of a_{ij} for any i.

(b) The dimension of v_i is the ratio of the dimension of c_j divided by the dimension of a_{ij} for any j.

In working with dimensions we use the rules of ordinary algebra for canceling and so on, as explained earlier in Section 3.

EXAMPLE 1 Let us return to the auto/truck example; its data box is given in Figure 6. We have already found the dimensions of the primal variables x_1 (trucks/week) and x_2 (autos/week). Let us use rule (b) above to determine the dimensions of the dual variables v_1 and v_2. For v_1 we have

$$\begin{aligned}\text{dimension of } v_1 &= (\text{dimension of } c_1)/(\text{dimension } a_{11})\\ &= \frac{\$}{\text{truck}}\Big/\frac{\text{S1-manhr}}{\text{truck}}\\ &= \frac{\$}{\text{truck}}\cdot\frac{\text{truck}}{\text{S1-manhr}}\\ &= \frac{\$}{\text{S1-manhr}}.\end{aligned}$$

In Exercise 1 the reader is asked to show that we would have obtained the same result if we had divided the dimension of c_2 by the dimension of a_{12}. In the same manner we have

$$\begin{aligned}\text{dimension of } v_2 &= (\text{dimension of } c_1)/(\text{dimension } a_{21})\\ &= \frac{\$}{\text{truck}} \cdot \frac{\text{truck}}{\text{S2-manhr}}\\ &= \frac{\$}{\text{S2-manhr}}.\end{aligned}$$

Figure 31 summarizes the complete data box for the auto/truck example, indicating the dimensions of all variables and constants.

	$x_1 \ \frac{\text{trucks}}{\text{week}}$	$x_2 \ \frac{\text{autos}}{\text{week}}$	Capacities
$v_1 \ \frac{\$}{\text{S1-manhr}}$	$5 \ \frac{\text{S1-manhr}}{\text{truck}}$	$2 \ \frac{\text{S1-manhr}}{\text{auto}}$	$180 \ \frac{\text{S1-manhr}}{\text{week}}$
$v_2 \ \frac{\$}{\text{S2-manhr}}$	$3 \ \frac{\text{S2-manhr}}{\text{truck}}$	$3 \ \frac{\text{S2-manhr}}{\text{auto}}$	$135 \ \frac{\text{S2-manhr}}{\text{week}}$
Profits	$300 \ \frac{\$}{\text{truck}}$	$200 \ \frac{\$}{\text{auto}}$	

Figure 31

	$x_1 \ \frac{\$}{\text{ton-HG}}$	$x_2 \ \frac{\$}{\text{ton-MG}}$	$x_3 \ \frac{\$}{\text{ton-LG}}$	Costs
$v_1 \ \frac{\text{M1-days}}{\text{week}}$	$6 \ \frac{\text{tons-HG}}{\text{M1-day}}$	$2 \ \frac{\text{tons-MG}}{\text{M1-day}}$	$4 \ \frac{\text{tons-LG}}{\text{M1-day}}$	$200 \ \frac{\$}{\text{M1-day}}$
$v_2 \ \frac{\text{M2-days}}{\text{week}}$	$2 \ \frac{\text{tons-HG}}{\text{M2-day}}$	$2 \ \frac{\text{tons-MG}}{\text{M2-day}}$	$12 \ \frac{\text{tons-LG}}{\text{M2-day}}$	$160 \ \frac{\$}{\text{M2-day}}$
Requirements	$12 \ \frac{\text{tons-HG}}{\text{week}}$	$8 \ \frac{\text{tons-MG}}{\text{week}}$	$24 \ \frac{\text{tons-LG}}{\text{week}}$	

Figure 32

EXAMPLE 2 The data box for the mining example is given in Figure 9. We already know that the dimensions of v_1 and v_2 are mine 1-days/week and mine 2-days/week, respectively. Let us use rule (a) above to find the dimensions of x_1.

$$\begin{aligned}\text{dimension of } x_1 &= (\text{dimension of } b_1)/(\text{dimension of } a_{11})\\ &= \frac{\$}{\text{M1-day}} \Big/ \frac{\text{tons-Hg}}{\text{M1-day}}\\ &= \frac{\$}{\text{tons-Hg}}.\end{aligned}$$

A similar application of rule (a) gives the dimensions of x_2 and x_3 as \$/ton-Mg and \$/ton-LG, respectively.

Figure 32 shows the data box for the mining example, indicating dimensions for all variables and constants.

Determining the dimensions of the dual variables is the first step in their interpretation. The next step is to look at the optimal dual solutions for the examples above and give their interpretations.

EXAMPLE 1 (continued) In Example 1 of Section 5 we found the optimal solution to the auto/truck example to be

$$x^0 = \begin{pmatrix} 30 \\ 15 \end{pmatrix}, \qquad v^0 = (\tfrac{100}{3}, \tfrac{400}{9}), \qquad f^0 = g^0 = 12{,}000.$$

We know that $v_1^0 = \frac{100}{3}$ has dimensions \$/S1-manhr, which sound like a *value* for shop 1 man-hours. We shall show that this is in fact the case. Suppose we increase the number of shop 1 man-hours from 180 to 183. Our problem is then summarized in the data box of Figure 33, where the dimensions

	x_1	x_2	Capacities
v_1	5	2	183
v_2	3	3	135
Profits	300	200	

Figure 33

are the same as in Figure 31 and are therefore omitted. The reader will be asked to show in Exercise 2 that the optimal solution to this problem is

$$x^0 = \begin{pmatrix} 31 \\ 14 \end{pmatrix} \quad \text{and} \quad v^0 = (\tfrac{100}{3}, \tfrac{400}{9}),$$

with objective value 12,100. Notice that the objective value has increased by 100, which is just three times the dual variable $v_1^0 = \frac{100}{3}$. Hence we see that $v_1^0 = 33.33$ is the *imputed value* of an additional hour of shop 1 man-hours. It should be remarked right away that the imputed-value interpretation holds over only a limited range of changes in shop 1 man-hours. Hence we should more properly say that $v_1^0 = 33.33$ is the *imputed value* of an additional hour in shop 1 *provided the dual solution is not changed by* adding this extra capacity.

Note also that the imputed value is determined independently of the *cost* of providing the extra man-hours in shop 1. In order to provide extra man-hours it would be necessary to pay workers overtime and rent additional equipment, or else do subcontracting, or the like. What the optimal dual variables tell us is the cost of providing extra hours in shop 1 should not be more than their imputed value, or else it is not optimal to get them.

In Exercise 3 the reader will be asked to show that the optimal dual variable $v_2^0 = \frac{400}{9} = 44.44$, which has dimensions \$ per shop 2 man-hour, is the imputed value of an additional hour in shop 2 provided the optimal dual solution does not change after the extra time is added. As before, it is the maximum amount one should be willing to pay to obtain the extra time.

EXAMPLE 2 (continued) In Example 2 of Section 5 we found the optimal solutions to the mining example to be

$$v^0 = (1, 3), \qquad x^0 = \begin{pmatrix} 10 \\ 70 \\ 0 \end{pmatrix}, \quad \text{and} \quad f^0 = g^0 = 680.$$

We know that $x_1^0 = 10$ has dimensions \$ per ton of high-grade ore, which sounds like the *imputed cost* of producing an additional ton of high-grade ore, and we shall show that this is the case. Suppose we increase the requirements for high-grade ore production from 12 to 16 tons. The new data box is shown in Figure 34, the dimensions being the same as in Figure 32.

	x_1	x_2	x_3	Costs
v_1	6	2	4	200
v_2	2	2	12	160
Requirements	16	8	24	

Figure 34

In Exercise 4 the reader will be asked to show that the optimal solution to the new problem is

$$v^0 = (2, 2), \qquad x^0 = \begin{pmatrix} 10 \\ 70 \\ 0 \end{pmatrix}, \quad \text{and} \quad f^0 = g^0 = 720.$$

Notice that the costs of production have increased from 680 to 720, which is $4 \cdot x_1^0 = 4 \cdot 10 = 40$. Hence $x_1^0 = 10$ was the per-ton cost of each of the additional 4 units of high-grade ore.

In Exercise 5 you will be asked to show that x_2^0 can be similarly interpreted as the *imputed* or *marginal cost* of producing an additional ton of medium-grade ore, *provided* the additional production does *not* cause a new dual solution to appear.

Now let us look at $x_3^0 = 0$, which has dimension \$ per ton of low-grade ore. What this says is that low-grade ore is free in the sense that producing an additional ton has zero cost. What does this mean? If we look at the slack vector $u^0 = (0, 0, 16)$ found in Section 5, we observe that there is an over-production of low-grade ore by 16 tons beyond the requirements. In other words we have already overproduced, so the additional ton will cost zero to produce since it already exists. However, this is true only within

	x_1	x_2	x_3	Costs
v_1	6	2	4	200
v_2	2	2	12	160
Requirements	12	8	56	

Figure 35

limits. For suppose we change the requirement for low-grade ore to 56 tons, giving the data box of Figure 35. In Exercise 6 the reader will be asked to show that the optimal solution to the problem in Figure 35 is

$$v^0 = (.5, 4.5), \qquad x^0 = \begin{pmatrix} 27.5 \\ 0 \\ 8.75 \end{pmatrix}, \quad \text{and} \quad f^0 = g^0 = 820.$$

Note that we now have a new dual solution, so that the old dual variable $x_3 = 0$ did not hold for the entire range of changes in the requirements for low-grade ore.

Let us try to give general interpretations to a pair of dual linear programming problems. For either problem the matrix A will be called the matrix of technological coefficients, since it indicates how activity vectors are combined into the constraining inequalities. Then we can give different interpretations to the vectors c, b, x, and v, depending on whether our original problem is a maximizing or a minimizing one.

If the original problem is maximizing, we interpret x as the *activity vector*. Then the vector b is interpreted as the *capacity-constraint vector*, whose components give the amounts of the various "scarce resources" that can be demanded by a given activity vector. The vector c is the *profit* vector, whose entries give the unit profits for each component of the activity vector x. Finally, the vector v is the *imputed-value vector,* whose entries give the imputed values of each of the scarce resources that enter into the production process, provided the changes in scarce resources are sufficiently small that the dual solution remains optimal.

If the original problem is minimizing, we interpret v as the *activity vector*. Then c is interpreted as the *requirements vector*, whose components give the minimum amounts of each good that must be produced. The vector b is the *cost vector*, whose entries give the unit costs of each of the activities. Finally, the vector x is the *imputed-cost* vector, whose components give the imputed costs of producing additional amounts of each of the required goods, *provided* the changes in requirements are sufficiently small that the dual solution remains optimal.

Next we shall briefly discuss two important theorems in linear programming. First we restate the dual problems:

The MAXIMUM Problem		*The MINIMUM Problem*	
Maximize	$cx = f$	Minimize	$vb = g$
subject to	(1) $Ax + y = b$,	subject to	(3) $vA - u = c$,
	(2) $x \geq 0, y \geq 0$.		(4) $v \geq 0, u \geq 0$.

Vectors x and y satisfying (1) and (2) and vectors v and u satisfying (3) and (4) are called *feasible vectors.*

In all the examples solved above we found that $f = g$ at the optimum solution. It is no accident that the dual problems share common values. The next theorem, which is the principal theorem of linear programming, shows that this will always happen whenever the problems have solutions.

The Duality Theorem The maximum problem has as a solution a feasible vector x^0, such that $cx^0 = \max cx$, if and only if the minimum problem has a solution that is a feasible vector v^0, such that $v^0 b = \min vb$. Moreover, the equality $cx^0 = v^0 b$ holds if and only if x^0 and v^0 are solutions to their respective problems.

The duality theorem is extremely powerful, for it says that if one of the problems has a (finite) solution, then the other one necessarily also has a (finite) solution, and both problems share a common value. Another consequence of the theorem is that if one of the problems does *not* have a solution, then neither does the other.

The proof of the duality theorem is beyond the scope of this book, but some parts of it are indicated in Exercises 25 and 26, and in Exercise 8 of Section 4. We saw an example of a linear programming problem without a solution in Example 4 in Section 5. Another example is in Exercise 27.

The duality theorem states that $g^0 = f^0$ at the optimum solution. Applying this to Tucker's duality equation [(7) in Section 4], we obtain:

$$0 = g^0 - f^0 = v^0 y^0 + u^0 x^0. \tag{5}$$

However, since v^0, y^0, u^0, and x^0 are all feasible optimal vectors, they are, in particular, nonnegative. Hence $v^0 y^0 \geq 0$ and $u^0 x^0 \geq 0$. But the only way that two nonnegative numbers can add up to zero is for both of them to be zero. Therefore

$$v^0 y^0 = 0, \tag{6}$$

$$u^0 x^0 = 0. \tag{7}$$

If we now simply restate (6) and (7), we obtain the following important theorem:

The Complementary Slackness Theorem

(A) For each i, *either* $v_i^0 = 0$ *or* $y_i^0 = b_i - \sum_{j=1}^{n} a_{ij}x_j^0 = 0$.

(B) For each j, *either* $x_j^0 = 0$ *or* $u_j^0 = \sum_{i=1}^{m} v_i^0 a_{ij} - c_j = 0$.

Proof The proof of this theorem is simple because (6) says that the sum of the products $v_i^0 y_i^0$ must equal zero, but each term of the product is nonnegative so each product must itself be zero, which gives (A). The proof of (B) follows similarly from (7).

EXAMPLE 2 (continued) From the final tableau in Figure 24 of the previous section we found that the complete solution to the mining problem to be

$$v^0 = (1, 3), \qquad u^0 = (0, 0, 16), \qquad x^0 = \begin{pmatrix} 10 \\ 70 \\ 0 \end{pmatrix}, \qquad y^0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

We see that since $u_3^0 = 16$—that is, in the optimal solution low-grade ore is overproduced—the imputed cost of low-grade ore must be zero; and it is, since $x_3^0 = 0$. Also, since both v_1^0 and v_2^0 are positive, both components of y^0 must be zero, which they are. The reader should state the other consequences of the complementary slackness theorem for this example.

Let us conclude by discussing the reasons for the various steps of the simplex method. If we always think of the nonbasic variables, which appear at the left and on the top of the tableaus (see Figures 14 and 17), as being set equal to zero, then in the initial tableau of Figure 14 we see the initial solution vectors

$$(8) \qquad x = 0, \qquad y = b, \qquad v = 0, \quad \text{and} \quad u = -c.$$

Since we have assumed $b \geq 0$, we see that the first three vectors are nonnegative, but u is nonnegative only if c was initially nonpositive. In the latter case the initial tableau is optimal (see Exercise 11). Since this is not normally the case, there is usually at least one positive indicator, so that the first answer to the question in box 2 of Figure 18 is "yes." Thus we must go around the loop and carry out at least one pivot. As we do so, the simplex method systematically changes the tableau in order to make u into a nonnegative vector without destroying the nonnegativeness of x, y, or v, and also keeping $f = cx = vb = g$ at all times.

In step 6 of Figure 18 the pivot was chosen in order to have the smallest ratio $t_{I,n+1}/t_{IJ}$ so that no current x_i or y_j should become negative. The reader may verify that if the pivot is chosen not to have this property, then some such variable is made negative (see Exercise 15 of the preceding section).

The nondegeneracy assumption made in Section 4 can be used to show (see Exercise 25) that on each pivot step the value of the current f will actually increase. In Exercise 26 you will be asked to show that at most a finite number of pivot steps can be made. Hence, if the problem has a solution, we must arrive in a finite number of steps at a tableau having all positive indicators. At each step the current solution in a tableau satisfies equations (1), (2), and (3) above, and when all indicators are positive we have also satisfied (4), so that $v \geq 0$ and $u \geq 0$. By the duality theorem, if we have found x^0, y^0, v^0, and u^0 satisfying (1)–(4) and also $f^0 = cx^0 = v^0 b = g^0$, then an optimum solution to the programming problem has been found.

EXERCISES

1. In Example 1 show that the same answer for the dimension of v_1 can be obtained by dividing the dimension of c_2 by the dimension of a_{12}.
2. Show that the vectors

$$x^0 = \begin{pmatrix} 31 \\ 14 \end{pmatrix} \quad \text{and} \quad v^0 = (\tfrac{100}{3}, \tfrac{400}{9})$$

solve the problem in Figure 33. [*Hint:* Substitute into the primal and dual problems.]

3. (a) Use the optimal solution to the automobile/truck problem in Figure 31 to predict how the objective function, which measures profit, will change if the capacity of shop 2 is changed from 135 to 144 man-hours per week.
   (b) Solve the problem in Figure 31 with the 135 changed to 144 and use its solution to show that your prediction in (a) was correct.

   [*Ans.* Profit 12,400, $x^0 = \begin{pmatrix} 28 \\ 20 \end{pmatrix}$, $v^0 = (\tfrac{100}{3}, \tfrac{400}{9})$.]
4. Show that the solution to the mining example in Figure 34 is

$$v = (2, 2), \qquad x^0 \text{ as before}, \qquad f = g = 720.$$

5. (a) Use the solution to Exercise 4 to predict what will happen in the mining problem if the requirement for medium-grade ore is increased from 8 to 10.
   (b) Solve the mining problem in Figure 34 with the 8 replaced by 10 and show that your prediction in (a) was correct.

   [*Ans.* $v^0 = (1.5, 3.5)$, x^0 as before, $f = g = 860$.]
6. Show that the solution to the problem in Figure 35 is

$$v^0 = (.5, 4.5), \qquad x^0 = \begin{pmatrix} 27.5 \\ 0 \\ 8.75 \end{pmatrix}, \qquad f = g = 820.$$

Interpret the solution.

7. In the automobile/truck example of Figure 31, suppose that the manufacturer can subcontract up to 18 of either shop 1 or shop 2 man-hours at $38 per hour. What is his optimal action? [*Hint:* You can answer this question without solving a linear programming problem.]

8. In the mining example of Figure 32 suppose the mining owner can sell 10 more tons of medium-grade ore at $55 per ton. Should he do so?

9. Consider again the general interpretation of a maximizing problem in which x is an activity vector, b the capacity-constraint vector, and c the profit vector. Let v^0 be the optimum dual solution vector. Discuss the following managerial interpretation of the components v_i^0 of v^0. "Additional amounts of scarce resource i should be acquired only if its cost is less than the component v_i^0 that gives the imputed value of an additional (sufficiently small) quantity."

10. Consider again the general interpretation of a minimizing problem in which v is the activity vector, c the requirements vector, and b the cost vector. Let x^0 be the optimum dual solution vector. Discuss the following managerial interpretations of the components x_i^0 of x. "Additional amounts of the jth good should be produced only if they can be sold with gross profit at least as large as the component x_j^0, which gives the imputed cost of producing an additional (sufficiently small) quantity."

11. Consider the dual maximum and minimum problems in equality form as expressed above. If $c \leq 0$, prove that the intial solution (8) is optimal. [*Hint:* Use the duality theorem.]

12–20. For each of Exercises 1–9 of Section 3 carry out the following steps:
- **(a)** Find the dimensions of the dual variables.
- **(b)** Set up the initial tableau with the dimensions of all variables and numbers indicated.
- **(c)** Read the answers to both primal and dual problems from the final tableau.
- **(d)** Interpret the dual solutions for the specific problems in each case.
- **(e)** State the complementary slackness theorem for each problem and interpret.

21–24. Rework Exercises 11–13 of Section 5 using steps (a)–(e) of Exercises 12–20, above.

***25.** The assumption of nondegeneracy stated in Section 5 can be shown to be equivalent to the following: At no time in the pivoting process of the simplex method are any of the entries in the first m rows of the last column of the tableau ever zero. Use this fact to show that on each pivot step the value of $f = cx$ *increases*.

***26.** Show that there are only a finite number of ways that the components of the x- and y-vectors can be used to label the top and right-hand side of the various tableaus during the pivoting process.

Use the result of Exercise 25 to show that no tableau can ever be repeated in the course of solving a nondegenerate problem by the simplex method. Hence, conclude that the simplex method described in Figure 18 must stop in a finite number of steps with the optimal solution to the linear programming problem, or else with proof that the problem has no finite solution.

27. Use the flow diagram of Figure 18 to show that the problem whose initial tableau is

-1	1	4
2	-4	8
2	3	0

does not have a solution. Verify algebraically and geometrically the statements in box 5 of that flow diagram.

*7 EQUALITY CONSTRAINTS AND THE GENERAL SIMPLEX METHOD

In this (optional) section we shall discuss the removal of the nonnegativity and nondegeneracy assumptions that we imposed at the beginning of Section 5 on linear programming problems. As stated there, most problems will automatically satisfy these assumptions. If not, they can usually be changed so that they do. We illustrate the latter first.

EXAMPLE 1 Consider again the automobile/truck example of Figure 6. Suppose we add the managerial constraint that at least 20 automobiles should be produced —perhaps because we have orders for them. The inequality that will do this is $x_2 \geq 20$, but it is a $\geq$ inequality instead of a $\leq$ inequality as is required for a maximizing problem. Multiplying through by -1 gives $-x_3 \leq -20$. Hence the maximizing problem is

$$(1)\qquad \begin{aligned} \text{Maximize}\quad & 300x_1 + 200x_2 \\ \text{subject to}\quad & 5x_1 + 2x_2 \leq 180, \\ & 3x_1 + 3x_2 \leq 135, \\ & -x_2 \leq -20, \\ & x_1, x_2 \geq 0. \end{aligned}$$

We see that the b vector is

$$(2)\qquad b = \begin{pmatrix} 180 \\ 135 \\ -20 \end{pmatrix},$$

which does not satisfy the nonnegativity assumption. However, let us set up the initial tableau and see what we can do with it. It is shown in Figure 36. Notice that in the third row where the -20 entry is, there is also a -1.

$$
\begin{array}{c|cc|c|l}
 & x_1 & x_2 & -1 & \\
\hline
v_1 & 5 & 2 & 180 & = -y_1 \\
v_2 & 3 & 3 & 135 & = -y_2 \\
v_3 & 0 & -1 & -20 & = -y_3 \\
\hline
-1 & 300 & 200 & 0 & = f \\
\hline
 & = u_1 & = u_2 & = g &
\end{array}
$$

Figure 36

If we were to pivot on the -1, using the usual rules as given in Figure 18, we could change the -20 into a $+20$. Carrying out this pivot operation gives the tableau of Figure 37, which *has* a positive b vector. Hence we

$$
\begin{array}{c|cc|c|l}
 & x_1 & y_3 & -1 & \\
\hline
v_1 & 5 & 2 & 140 & = -y_1 \\
v_2 & 3 & 3 & 75 & = -y_2 \\
v_3 & 0 & 1 & 20 & = -x_2 \\
\hline
-1 & 300 & 200 & -4000 & = f \\
\hline
 & = u_1 & = v_3 & = g &
\end{array}
$$

Figure 37

can now proceed in the usual way. Choosing the most positive indicator, which is 300, we determine that the pivot should be the 3 circled in the first column. Carrying out the rest of the pivot steps as in Figure 18 gives the tableau in Figure 38. Since all indicators there are negative, we have

$$
\begin{array}{c|cc|c|l}
 & y_2 & y_3 & -1 & \\
\hline
v_1 & -\frac{5}{3} & -3 & 65 & = -y_1 \\
u_1 & \frac{1}{3} & 1 & 25 & = -x_1 \\
u_2 & 0 & 1 & 20 & = -x_2 \\
\hline
-1 & -100 & -100 & -11{,}500 & = f \\
\hline
 & = v_2 & = v_3 & = g &
\end{array}
$$

Figure 38

determined the optimal solution, namely

$$x^0 = \begin{pmatrix} 25 \\ 20 \end{pmatrix}, \qquad v^0 = (0, 100, 100), \quad \text{and} \quad f^0 = g^0 = 11{,}500.$$

In other words, the optimum decision now is to produce 25 trucks and 20 automobiles for a gross profit of \$11,500. Notice that the gross profit has gone down, which is not surprising since we are satisfying an additional constraint. Notice also that the dual solution indicates that for each automobile less that we require to be made, an additional \$100 profit can be realized. This follows because $v_3^0 = \$100$, indicating that if we increase the right-hand side of the third constraint by 1, that is, change -20 to -19, then the profit should increase by \$100. Notice also that the imputed value of shop 1 man-hours has gone to zero! This is because $y_1 = 65$, indicating that we are not using all of the shop 1 man-hours. Also the imputed value of shop 2 man-hours has jumped from \$44.44 to \$100 per hour, which indicates that shop 2 has become a more important "bottleneck" in the production process.

The previous example shows one way of deriving a problem that has negative b vector coefficients—namely, by imposing a $\leq$ constraint with positive right-hand side on the maximizing problem. Another way is to impose an equality constraint. For example, consider the equation

$$2x_1 + 5x_2 - 7x_3 = 12. \tag{3}$$

We can replace it by the two inequalities

$$2x_1 + 5x_2 - 7x_3 \leq 12 \quad \text{and} \quad 2x_1 + 5x_2 - 7x_3 \geq 12, \tag{4}$$

but the second of these is a $\geq$ constraint. We can change it into a $\leq$ constraint by multiplying by a -1, obtaining

$$2x_1 + 5x_2 - 7x_3 \leq 12 \quad \text{and} \quad -2x_1 - 5x_2 + 7x_3 \leq -12 \tag{5}$$

as a pair of $\leq$ inequalities that are equivalent to the single equality (3).

When solving simple problems such as in Example 1 by hand it is usually quite easy to see how to pivot on negative numbers in the tableau in such a way that the problem becomes one having nonnegative right-hand sides. However, for large problems, and particularly for computing-machine computation, it is necessary to have a set of rules that will always work, without depending upon the ingenuity of the user. Such an algorithm is presented in Figure 39. It is usually called "phase I" of the simplex method, and what it does is to put the tableau in the standard form so that the flow diagram of Figure 18 can be applied. We illustrate it with an example.

EXAMPLE 2 Consider the linear programming problem

$$\begin{array}{ll} \text{Maximize} & 2x_1 + x_2 \\ \text{subject to} & x_1 + x_2 \leq 20, \\ & x_1 + 2x_2 = 30, \\ & x_1, x_2 \geq 0. \end{array} \tag{6}$$

The set of feasible x-vectors is the line segment between the points $\begin{pmatrix} 0 \\ 15 \end{pmatrix}$ and $\begin{pmatrix} 10 \\ 10 \end{pmatrix}$ shown darkened in Figure 40. In order to solve (6) we replace the equality constraint by a pair of inequalities and obtain the problem:

$$\begin{array}{ll} \text{Maximize} & 2x_1 + x_2 \\ \text{subject to} & x_1 + x_2 \leq 20, \\ & x_1 + 2x_2 \leq 30, \\ & -x_1 - 2x_2 \leq -30, \\ & x_1, x_2 \geq 0. \end{array} \tag{7}$$

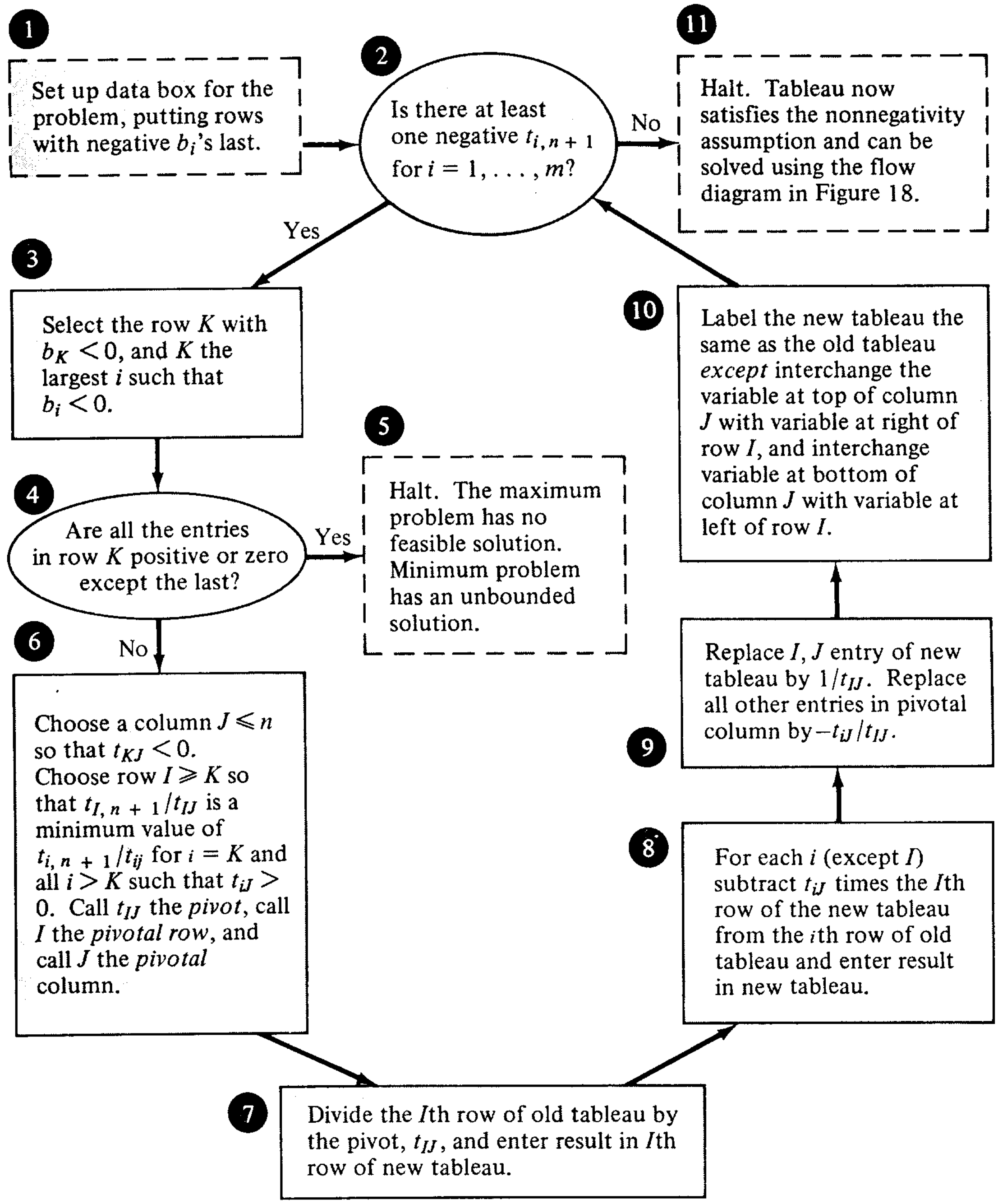


Figure 39

Thus we obtain a problem that does not satisfy the nonnegativity assumption.

Let us solve the problem by following the flow diagram of Figure 39. We set up the initial tableau with the negative b_i's last as instructed in box 1 of that figure. The initial tableau is given in Figure 41. The answer to the question in box 2 of Figure 39 is "yes," so we go to box 3, where we must choose $K = 3$. The answer to the question in box 4 is "no," so we go on to box 6. Since both entries in the first two columns of the third row of Figure 41 are negative, J can be either 1 or 2; we choose $J = 1$. Then the ratio rule in box 6 gives $I = 3$. Carrying out the pivot steps in boxes 7–10 of Figure 39 gives the next tableau shown in Figure 42. Notice

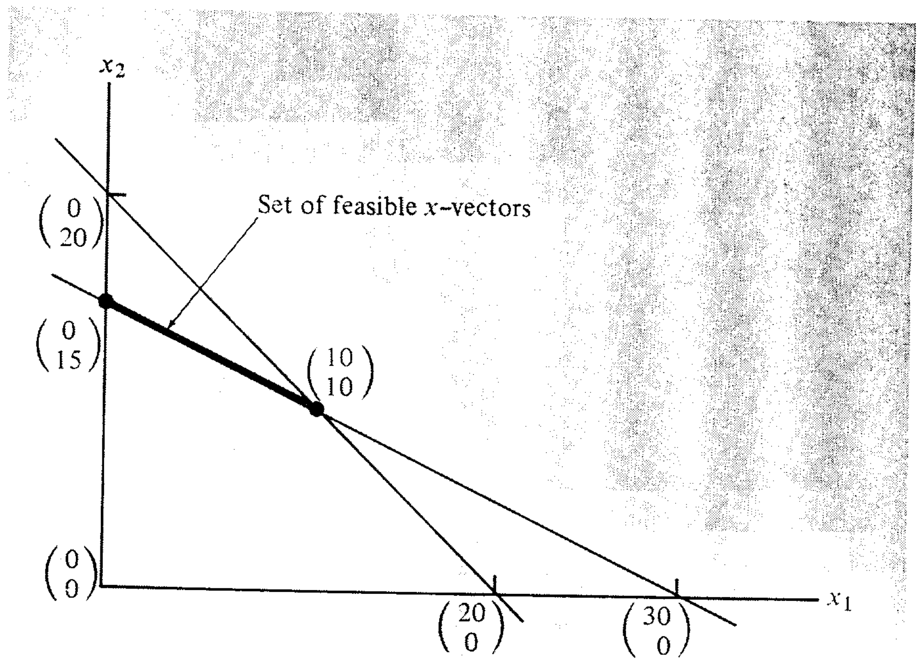


Figure 40

that a new negative has appeared in the third column of the first row! So the answer to the question in box 2 of Figure 39 is again "yes," and we must go around the main loop of the flow diagram again. We find that

	x_1	x_2	-1	
v_1	1	1	20	$= -y_1$
v_2	1	2	30	$= -y_2$
v_3	-1	-2	-30	$= -y_3$
-1	2	1	0	$= f$
	$= u_1$	$= u_2$	$= g$	

Figure 41

$K = 1$ and $J = 2$ are the only possible choices, and these give $I = 1$, so that we must pivot on the -1 circled in the first row of Figure 42. After

	y_3	x_2	-1	
v_1	1	-1	-10	$= -y_1$
v_2	1	0	0	$= -y_2$
u_1	-1	2	30	$= -x_1$
-1	2	-3	-60	$= f$
	$= v_3$	$= u_2$	$= g$	

Figure 42

pivoting, the new tableau is as shown in Figure 43. Since both indicators are negative, we have obtained the optimal solution without further pivoting.

	y_3	y_1	-1	
u_2	-1	-1	10	$= -x_2$
v_2	1	0	0	$= -y_2$
u_1	1	2	10	$= -x_1$
-1	-1	-3	-30	$= f$
	$= v_2$	$= v_1$	$= g$	

Figure 43

It is

$$x^0 = \begin{pmatrix} 10 \\ 10 \end{pmatrix}, \qquad v^0 = (3, 0, 1), \quad \text{and} \quad f^0 = g^0 = 30.$$

The reader should locate the solution on the diagram of Figure 40.

The last topic of this section is the question of removing the nondegeneracy assumption stated in Section 5. A complete discussion of the problem is beyond the scope of this book, but an interested reader may wish to refer to one of the more advanced texts listed at the end of this chapter. We shall indicate the essential ideas here, however. An example will suffice for this purpose.

EXAMPLE 3 Consider the problem:

$$(8) \qquad \begin{array}{ll} \text{Maximize} & x_1 + x_2 \\ \text{subject to} & x_1 \phantom{{}+x_2} \leq 4, \\ & \phantom{x_1 + {}} x_2 \leq 4, \\ & 2x_1 + x_2 \leq 8, \\ & x_1, x_2 \geq 0. \end{array}$$

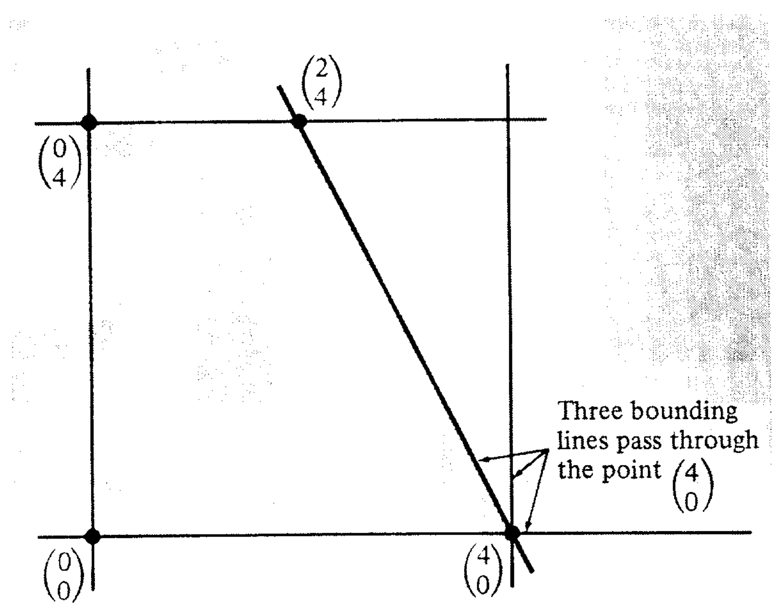


Figure 44

The set of feasible x-vectors is shown shaded in Figure 44. Notice that the set has four extreme points and that each is the intersection of exactly two bounding lines *except* for the point $\begin{pmatrix}4\\0\end{pmatrix}$, which has three bounding lines through it. We shall show that this can lead to the appearance of a zero in the b area of the tableau after some pivoting, and when this happens it is possible to pivot without improving the objective function. The initial tableau for the problem is given in Figure 45.

$$\begin{array}{c|cc|c|l} & x_1 & x_2 & -1 & \\ \hline v_1 & 1 & 0 & 4 & = -y_1 \\ v_2 & 0 & 1 & 4 & = -y_2 \\ v_3 & 2 & 1 & 8 & = -y_3 \\ \hline -1 & 1 & 1 & 0 & = g \\ \hline & = u_1 & = u_2 & = g & \end{array}$$

Figure 45

Since both indicators are positive, suppose we choose the first one. The minimum-ratio rule then selects the first row to be pivotal, and we pivot on the one circled. (Note that we could also pivot on the 2 in the third row, first column, and the results will be similar; see Exercise 11.) The new tableau is given in Figure 46. Notice that a zero did appear in the third

$$\begin{array}{c|cc|c|l} & y_1 & x_2 & -1 & \\ \hline u_1 & 1 & 0 & 4 & = -x_1 \\ v_2 & 0 & 1 & 4 & = -y_2 \\ v_3 & -2 & 1 & 0+\epsilon & = -y_3 \\ \hline -1 & -1 & 1 & -4 & = f \\ \hline & = v_1 & = u_2 & = g & \end{array}$$

Figure 46

row, third column, of Figure 46. In order to make it into something positive a small amount ϵ is added to it. This is called a *perturbation*. Geometrically it corresponds in Figure 44 to moving the line $2x_1 + x_2 = 8$ parallel to itself upward slightly. This makes the extreme point $\begin{pmatrix}4\\0\end{pmatrix}$ have just two bounding lines through it, and adds a new extreme point $\begin{pmatrix}4\\\epsilon\end{pmatrix}$ nearby. We will find it on the next iteration. The second column has a positive indicator, and the minimum-ratio rule selects the third row to be pivotal and 1 the pivot, circled in Figure 46. The new tableau is given in Figure 47.

Now we observe that column 1 has a positive indicator, so we must still

	y_1	y_3	-1	
u_1	1	0	4	$= -x_1$
v_2	2	-1	$4 - \epsilon$	$= -y_2$
u_2	-2	1	$0 + \epsilon$	$= -x_2$
-1	1	-1	$-4 - \epsilon$	$= f$
	$= v_1$	$= v_3$	$= g$	

Figure 47

	y_2	y_3	-1	
u_1	$-\frac{1}{2}$	$-\frac{1}{2}$	$2 + (\frac{\epsilon}{2})$	$= -x_1$
v_1	1	$-\frac{1}{2}$	$2 - (\frac{\epsilon}{2})$	$= -y_1$
u_2	-1	0	4	$= -x_2$
-1	$-\frac{1}{2}$	$-\frac{1}{2}$	$-6 - (\frac{\epsilon}{2})$	$= f$
	$= v_2$	$= v_3$	$= g$	

Figure 48

pivot again. The ratio rule selects as pivot the 2 in the first column, circled in Figure 47. The next tableau is given in Figure 48.

Since both indicators in Figure 48 are negative, we have the optimal solution. Notice that if we replace ϵ by 0 we still have an optimal tableau, hence our perturbation did not affect the original problem enough to change the solution, which is

$$x^0 = \begin{pmatrix} 2 \\ 4 \end{pmatrix}, \qquad v^0 = (0, \tfrac{1}{2}, \tfrac{1}{2}), \quad \text{and} \quad f^0 = g^0 = 6.$$

Actually, if we had ignored the 0 in the last column of Figure 47 and just gone ahead with the simplex method as given in Figure 18, we would have arrived at the same solution without difficulty. But notice that in going from tableau 46 to 47 we then would not have increased the objective function f at all. It can happen with larger problems that the computation could go from one tableau to the next several times in a row without changing f, and after a finite number of pivots return to a tableau constructed earlier. From then on the computational process will go through the same sequence of tableaus indefinitely without changing f. This is called *cycling*. Actually it rarely happens in practice. The smallest known example in which it can occur has seven variables. For small problems that can be worked by hand it never occurs.

There are several ways of avoiding cycling for computer codes that handle large problems. One way is the process of perturbation illustrated above. There only one 0 was found and it was made positive by adding $+\epsilon$ to it. If a second zero were found, then $+\epsilon^2$ would be added; and if a third were found, then $+\epsilon^3$ would be added; and so on. The final tableau will then have numbers plus polynomials in ϵ in the last column. By selecting

ϵ *not* to be equal to any of the finite number of zeros of these polynomials and also very small, we can prove that there always is a perturbation of the components of the b-vectors that will avoid cycling, and that has the same solution as the original problem when ϵ is replaced by 0 in the final tableau.

Still another (practical) way of avoiding cycling is the following. Whenever a zero is about to appear in a tableau, there will be more than one choice of pivotal row in box 6 of the flow diagram of Figure 18. This can be seen in Figure 45, in which, given the pivotal column $J = 1$, we can choose *either* $I = 1$ *or* $I = 3$ when applying the test. Suppose now we choose between these two at random, instead of always choosing the first one. It can be shown that if this method is used to "break ties" when selecting pivotal rows, then the simplex method will not cycle with probability 1. For practical purposes this provides an adequate safeguard against the very rare possibility of cycling in computations.

EXERCISES

1. Write pairs of $\leq$ inequalities that are equivalent to each of the following $=$ constraints:

(a) $12x_1 + 3x_2 - 7x_3 = 15$.

(b) $3x_1 - 2x_2 + 4x_3 = 0$ and $-4x_1 + x_2 - 2x_3 = 7$.

2. Consider the mining example (Example 2 of Section 3) again with the additional constraint that exactly 16 tons of high-grade ore should be produced per week. Show that the tableau has a nonnegative b-vector.

3. Show that a minimizing problem with $b \geq 0$ can always be solved using Figure 18 regardless of the form of the additional constraints that may be imposed on the minimizing problem.

4. In Example 1 of Section 3 show that the additional constraint $x_1 \leq 15$ can be imposed and the problem solved using Figure 18.

5. Show that a maximizing problem with only $\leq$ constraints and positive b-vector can be solved using Figure 18 regardless of how many additional $\leq$ constraints are added, as long as the right-hand sides of such additional constraints are nonnegative.

6. Use the results of Exercises 3–5 to show that the phase I computation of Figure 39 is needed only when a $\leq$ constraint with negative right-hand side is added to the maximizing problem.

7. Apply the phase I simplex method of Figure 39 to the following examples.

(a) Maximize $2x_1 + x_2$ subject to

$$\begin{aligned} x_1 + x_2 &\leq 10, \\ x_1 + x_2 &\geq 6, \\ x_1 &\leq 8, \\ x_1, x_2 &\geq 0. \end{aligned}$$

(b) Maximize x_1 subject to

$$\begin{aligned} x_1 &\geq 2, \\ x_2 &\geq 3, \\ 3x_1 + 2x_2 &\leq 24. \end{aligned}$$

8. Apply the phase I computation to the problem whose initial tableau is given by

1	1	20
−1	−2	−50
2	1	0

and show that the computation ends up in box 5 of Figure 39. Draw the constraint sets of the primal and dual problems and give a geometric interpretation to the statements in box 5 of Figure 39.

*9. Show in general that if the computation of Figure 39 ends up in box 5, then the statements given there are correct.

*10. Show that phase I is needed if and only if $x = 0$ is *not* a feasible vector for the maximizing problem.

11. Start with Figure 45 and carry out pivoting steps, starting with the pivot in the third row, first column. Show that equivalent results are obtained.

12. Show that even if we do not add $+\epsilon$ in the third row, third column, of Figure 46, the simplex method will yield the correct solution.

13. Add the constraint $-x_1 + x_2 \leq 4$ to the problem in (8) and show that no matter which column is chosen for the first pivot, a 0 is still produced in the b-vector after one pivot. Show that the simplex method still works.

*14. (a) Show that the phase I simplex method will eventually make the last inequality with negative right-hand side into one with positive (or zero) right-hand side without making the right-hand sides of later inequalities negative.

(b) Show that in a finite number of steps all negative right-hand sides will be made nonnegative, or else the computation will end up in box 5 of the flow diagram in Figure 39.

8 STRICTLY DETERMINED GAMES

In Sections 1–7 we discussed linear programming problems that involve *optimization*—that is, the maximization or minimization of a (linear) function subject to linear constraints. In order to optimize a function it is necessary to control all relevant variables.

Game theory considers situations in which there are two (or more) persons, each of whom controls some but not all the variables necessary to determine the outcome(s) of a certain event. Depending upon which event actually occurs, the players receive various payments. If for each possible event the algebraic sum of payments to all players is zero, the game is called *zero-sum;* otherwise it is *nonzero-sum*. Usually the players will not agree as to which event should occur, so that their objectives in the game are different. In the case of a matrix game, which is a two-person zero-sum game in which one player loses what the other wins, game theory provides a solution. The solution is based on the principle that each player tries to choose his course of action so that, regardless of what his opponent does, he can assure himself of a certain minimum amount. Matrix games are discussed in Sections 8 through 11. We shall not discuss nonzero-sum games in this chapter. We refer an interested reader to the suggested readings at the end of the chapter for treatments of this important class of games.

Most recreational games such as ticktacktoe, checkers, backgammon, chess, poker, bridge, and other card or board games can be viewed as games of strategy. On the other hand, such gambling games as dice, roulette, and so on are not (as usually formulated) games of strategy, since a person playing one of these games is merely "betting against the odds."

In this and the following sections we shall formulate simple games that illustrate the theory and are amenable to computation. We shall base these games on applications in business situations and on recreational games.

EXAMPLE 1 Two stores, R and C, are planning to locate in one of two towns. As in Figure 49, town 1 has 60 percent of the population while town 2 has 40 percent. If both stores locate in the same town they will split the total business of both towns equally, but if they locate in different towns each will get the business of that town. Where should each store locate?

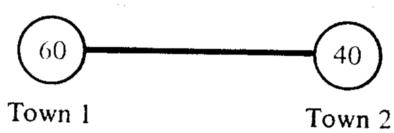


Figure 49

		Store C locates in 1	Store C locates in 2
Store R locates in	1	50	60
	2	40	50

Figure 50

Clearly this is a game situation, since each store can control where it locates but cannot control at all where its competitor locates. Each store has two possible "strategies": "locate in town 1" and "locate in town 2." Let us list all possible outcomes for each store employing each of its strategies. The result is given in the *payoff matrix* of Figure 50. The entries of the matrix represent the percentages of business that store R gets in each case. They can also be interpreted as the percentage *losses* of business by C for each case. If both stores locate in town 1 or both in town 2, each gets 50 percent of the business, hence the entries on the main diagonal are 50. If store *R* locates in town 1 and C in 2, then R gets 60 percent of the business as indicated in entry in row 1 and column 2. (This entry also indicates that C *loses* 60 percent.) Similarly, if R locates in 2 and C in 1, then R gets 40 percent (and C loses 40 percent) as indicated in row 2 and column 1.

How should the players play the matrix game in Figure 50? It is easy to see that store R should prefer to locate in town 1 because, regardless of what C does, R can assure himself of 10 percent more business in town 1 than in town 2. Similarly, store C also prefers to locate in town 1 because he will lose 10 percent less business—that is, gain 10 percent more business—in town 1 than in 2. Hence optimal strategies are for each store to locate in town 1; that is, R chooses row 1 and C chooses column 1 in Figure 50. The value of the game is 50, representing the percentage of the business that R gets.

In Example 1 we started with an applied situation and derived from it a matrix game. Actually, we can interpret any matrix as a game, as the following definition shows.

Definition Let G be an $m \times n$ matrix with entries g_{ij} for $i = 1, \ldots, m$ and $j = 1, \ldots, n$. Then G can be interpreted as the *payoff matrix* of the following matrix game: player R (the *row* player) chooses any row i, and simultaneously player C (the *column* player) chooses any column j; the outcome of the game is that C pays to R an amount equal to g_{ij}. (If $g_{ij} < 0$, then this should be interpreted as R paying C an amount equal to $-g_{ij}$.)

EXAMPLE 2 Consider the matrix in Figure 51 as a game. Thus, if R chooses row 1 and C chooses column 1, then C pays 5 units to R; if R chooses row 1 and C

		C Chooses 1	C Chooses 2
R Chooses	1	5	−10
	2	2	0

Figure 51

chooses column 2, then R pays 10 units to C; and so on. How should the players play this game?

Player R would like to get the 5 payoff, and is tempted to play row 1. However, player C clearly prefers to play column 2, since each entry in it is lower than the corresponding entry in column 1. And since player R realizes this, he will play row 2 to avoid the −10 payoff. The optimal strategies then are "play row 2" for R, and "play column 2" for C. The value of the game is $g_{22} = 0$.

The solutions in the first two examples have the following in common. In each case the value is an entry that is the minimum of its row and the maximum of its column. Such an entry is called a *saddle value.* When such a saddle value exists, it is always the value of the game, and the game is strictly determined. To see this, consider any game G with an entry $g_{ij} = v$ which is a saddle value. Then, since v is the minimum of row i, R can by playing row i assure that he will win at least v. And since v is the maximum of column j, C by playing column j can assure that R will not win more than v. This justifies the definition:

Definition Consider a matrix game with payoff matrix G. Entry g_{ij} is said to be a *saddle value* of G if g_{ij} is simultaneously the *minimum* of the ith row and the *maximum* of the jth column. If matrix game G has a saddle value, it is said to be *strictly determined*, and *optimal strategies* for it are:

For player R: "Choose a row that contains a saddle value."
For player C: "Choose a column that contains a saddle value."

The *value* of the game is $v = g_{ij}$, where g_{ij} is any saddle-value entry. The game is *fair* if its value is zero.

In order to justify this definition it must be shown that if there are two or more saddle values then they are all equal. A proof of this fact is outlined in Exercise 10. The next example illustrates it.

EXAMPLE 3 Let us consider an extension of Example 1 in which the stores R and C are trying to locate in one of the three towns in Figure 52. We shall assume that if both stores locate in the same town they split all business equally, but if they locate in different towns then all the business in the town that

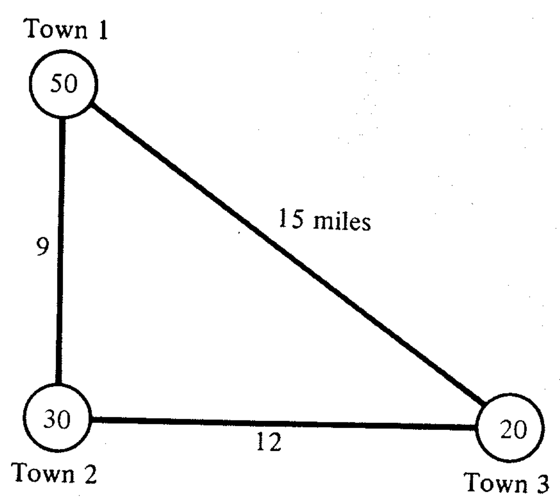


Figure 52

doesn't have a store will go to the *closer* of the two stores. The percentages of people in each town are marked in the circles. The distances between the towns are marked on the lines connecting them.

The payoff matrix for the resulting game is shown in Figure 53. In Exercise 13 the reader is asked to check that these entries are correct.

Each of the four 50 entries in the 2×2 matrix in the upper left-hand corner of Figure 53 is a saddle value of the matrix, since each is simultaneously the minimum of its row and maximum of its column. Note that

		Store C locates in 1	2	3
Store R locates in	1	50	50	80
	2	50	50	80
	3	20	20	50

Figure 53

the 50 entry in the lower right-hand corner is *not* a saddle value. Hence the game is strictly determined, and optimal strategies are:

For store R: "Locate in either town 1 or town 2."
For store C: "Locate in either town 1 or town 2."

In a real-life location problem one might want to take into account not only present populations of cities, but also rate of population growth. In Exercise 14 the reader is asked to criticize the above strategies from this point of view.

Instead of the somewhat indefinite description of the optimal strategy for player R as "Locate in either town 1 or 2," we can employ the following device: since we don't care which town we locate in, we can just flip a coin, or use any other chance device, and on the basis of the outcome make the choice between the towns. So we can also use the following strategy: "Select one of the numbers 1 or 2 by means of a random device with arbitrary probabilities for each outcome, and locate in the corresponding town." This strategy is also optimal.

Note that if we multiply the matrix in Figure 53 on the left by the vector $(1, 0, 0)$, we get the first row; hence we shall use this vector to represent the strategy "Locate in town 1" for store R. Similarly, the strategy "Locate in town 2" is represented by the vector $(0, 1, 0)$, since multiplying the matrix on the left by it gives the second row. Then the vector

$$(a, 1 - a, 0) = a(1, 0, 0) + (1 - a)(0, 1, 0) \quad \text{for} \quad 0 \leq a \leq 1$$

represents the strategy "Choose row 1 with probability a and row 2 with probability $1 - a$."

Similarly, for store C, the column vectors

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} a \\ 1 - a \\ 0 \end{pmatrix} \quad \text{for} \quad 0 \leq a \leq 1$$

represents the strategies "Locate in town 1," "Locate in town 2," and "Locate in town 1 with probability a and in town 2 with probability $1 - a$," respectively.

EXAMPLE 4 Consider the game G whose matrix is in Figure 54. It is not hard to see that the game is strictly determined with value 1, and there are four saddle

Player C

Player R	1	5	1	7
	−2	8	0	−9
	1	12	1	3

Figure 54

values. Optimal strategies are $(1, 0, 0)$ and $(0, 0, 1)$ for player R, and

$$\begin{pmatrix}1\\0\\0\\0\end{pmatrix} \text{ and } \begin{pmatrix}0\\0\\1\\0\end{pmatrix}$$

for player C. The four ways we can pair optimal strategies for player R with those for player C give the four saddle values. Besides the optimal strategies above we have their convex combinations

$$a(1, 0, 0) + (1 - a)(0, 0, 1) = (a, 0, 1 - a),$$

which is optimal for R for any a satisfying $0 \leq a \leq 1$, and

$$a\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + (1 - a)\begin{pmatrix}0\\1\\0\\0\end{pmatrix} = \begin{pmatrix}a\\0\\1 - a\\0\end{pmatrix},$$

which is optimal for player C for any a in the same range.

As the reader may have already found out for himself, not all matrix games are strictly determined. For instance, the two games shown in Figure 55 are not strictly determined. The solution of such games will be discussed in succeeding sections.

0	1
2	0

(a)

5	−2	3
−5	0	7
3	4	−1

(b)

Figure 55

EXERCISES

1. Determine which of the games given below are strictly determined and which are fair. When the game is strictly determined, find optimal strategies for each player.

(a)

0	2
−1	4

(b)

5	0
0	2

(c)

3	1
4	0

(d)

1	−1
−1	1

(e)

3	1
−4	0

(f)

0	4
0	2

(g)

7	0
0	0

(h)

0	0
0	−7

(i)

0	0
0	0

(j)

1	1
1	1

[*Partial Ans.* (a) Strictly determined and fair; R play row 1, C play column 1; (b) nonstrictly determined; (e) strictly determined but not fair; R play row 1, C play column 2; (j) strictly determined but not fair; both players can use any strategy.]

2. Find the value and all optimal strategies for the following games:

(a)

15	2	−3
6	5	7
−7	4	0

(b)

5	2	−1	−1
1	1	0	1
3	0	−3	7

(c)

0	5	6	−3
1	−1	2	3
1	2	3	4
−1	0	7	5

(d)

1	−12	6
0	−4	1
3	−7	2
3	−4	2
−5	−4	7

[*Ans.* (a) $v = 5$; $(0, 1, 0)$; $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$; (d) $(0, a, 0, 1 - a, 0)$, $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $v = -4$.]

3. Find the values of and all optimal strategies for the following games:

(a)

5	10	6	5
5	7	8	5
0	5	6	5

(b)

−2	0	−1
−5	7	8

(c)

0	0	1	0
1	0	0	0
1	0	1	0

(d)

3	2	3
6	2	7
5	1	4

[*Ans.* (a) $v = 5$; $(a, 1 - a, 0)$; $\begin{pmatrix} a \\ 0 \\ 0 \\ 1 - a \end{pmatrix}$; (d) $v = 2$; $(a, 1 - a, 0)$; $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.]

4. Each of two players shows one or two fingers (simultaneously) and C pays to R a sum equal to the total number of fingers shown. Write the game matrix. Show that the game is strictly determined, and find the value and optimal strategies.

5. Each of two players shows one or two fingers (simultaneously) and C pays to R an amount equal to the total number of fingers shown, while R pays to C an amount equal to the product of the numbers of fingers shown. Construct the game matrix (the entries will be the net gain of R), and find the value and the optimal strategies.

[*Ans.* $v = 1$, R must show one finger, C may show one or two.]

6. Show that a strictly determined game is fair if and only if there is a zero entry such that all entries in its row are nonnegative and all entries in its column are nonpositive.

7. Consider the game

$G =$

2	5
−1	a

(a) Show that G is strictly determined regardless of the value of a.

(b) Find the value of G. [*Ans.* 2.]

(c) Find optimal strategies for each player.

(d) If $a = 1{,}000{,}000$, obviously R would like to get it as his payoff. Is there any way he can assure himself of obtaining it? What would happen to him if he tried to obtain it?

(e) Show that the value of the game is the most that R can assure for himself.

8. Consider the matrix game

$$G = \begin{array}{|c|c|}\hline a & a \\ \hline c & d \\ \hline \end{array}$$

Show that G is strictly determined for every set of values for a, c, and d. Show that the same result is true if two entries in a given column are equal.

9. Find necessary and sufficient conditions that the game

$$G = \begin{array}{|c|c|}\hline a & 0 \\ \hline 0 & b \\ \hline \end{array}$$

should be strictly determined. [*Hint:* These will be expressed in terms of relations among the numbers a and b and the number zero.]

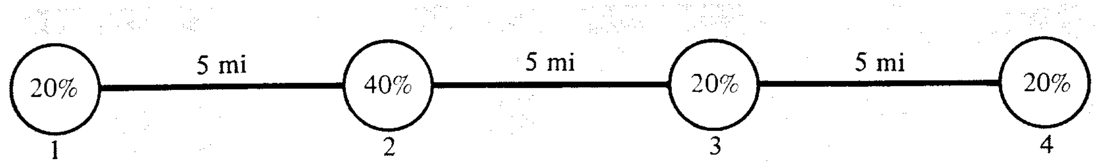
10. **(a)** Show that if there are two saddle values in the same row, then they are equal.

(b) Show that if there are two saddle values in the same column, then they are equal.

(c) If g_{ij} and g_{hk} are saddle values in different rows and columns, show that $g_{ij} = g_{ik}$. Also show $g_{ik} = g_{hk}$.

(d) Prove that $g_{ij} = g_{hk}$.

11. Two companies, one large and one small, manufacturing the same product, wish to build a new store in one of four towns located on a given highway. If we regard the total population of the four towns as 100 percent, the distribution of population and distances between towns are as shown:



Assume that if the large company's store is nearer a town, it will capture 80 percent of the business; if both stores are equally distant, then the large company will capture 60 percent of the business; and if the small store is nearer, then the large company will capture 40 percent of the business.

(a) Set up the matrix of the game.

(b) Test for dominated rows and columns, that is, rows or columns that will never be used by a player who plays optimally.

(c) Find optimal strategies and the value of the game and interpret your results.

[*Ans.* Both companies should locate in town 2; the large company captures 60 percent of the business.]

12. Rework Exercise 11 if the percentages of business captured by the large company are 90, 75, and 60, respectively.
13. Show that the entries in Figure 53 are correct.
14. In the store location of Example 3 how do the optimal strategies change if the population of town 1 becomes 51 percent and the population of town 2 becomes 29 percent of the total? How might they change if town 2 is growing much faster than town 1?
15. Show that the following game is always strictly determined for non-negative a and any values of the parameters b, c, d, and e.

$2a$	a	$3a$
b	$-a$	c
d	$-2a$	e

16. For what values of a is the following game strictly determined?

[*Ans.* $-1 \le a \le 2$.]

a	6	2
-1	a	-7
-2	4	a

9 MATRIX GAMES

As we saw in the numerical examples of the previous section, some matrix games are nonstrictly determined; that is, they have no entry that is simultaneously a row minimum and a column maximum. We can characterize nonstrictly determined 2×2 matrix games as follows:

Theorem The matrix game

$G =$

a	b
c	d

is nonstrictly determined if and only if one of the following two conditions is satisfied:

(i) $a < b, \quad a < c, \quad d < b, \quad$ and $\quad d < c$.
(ii) $a > b, \quad a > c, \quad d > b, \quad$ and $\quad d > c$.

(These equations mean that the two entries on one diagonal of the matrix must each be greater than each of the two entries on the other diagonal.)

Proof If either of the conditions (i) or (ii) holds, it is easy to check that no entry of the matrix is simultaneously the minimum of the row and the maximum of the column in which it occurs; hence the game is not strictly determined.

To prove the other half of the theorem, recall that, by Exercise 8 of the last section, if two of the entries in the same row or the same column of G are equal, the game is strictly determined; hence we can assume that no two entries in the same row or the same column are equal. Suppose now that $a < b$; then $a < c$ or else a is a row minimum and a column maximum; then also $c > d$ or else c is a row minimum and a column maximum; then also $d < b$ or else d is a row minimum and a column maximum. Hence the assumption $a < b$ leads to case (i) above.

In a similar manner the assumption $a > b$ leads to case (ii). This completes the proof of the theorem.

EXAMPLE 1 Jones and Smith play the following game: Jones conceals either a \$1 or a \$2 bill in his hand; Smith guesses 1 or 2, winning the bill if he guesses the number. If we make Jones player R (the row player) and Smith player C, the matrix of the game is as in Figure 56. Because the game satisfies condition (i) in the theorem above, the game is nonstrictly determined. Later we shall solve it.

		Player C Smith guesses 1	2
Player R Jones chooses	\$1 bill	−1	0
	\$2 bill	0	−2

Figure 56

EXAMPLE 2 Mr. Sub works for Mr. Super and frequently must advise him on the acceptability of certain projects. Whenever Mr. Sub can make a clear judgment about a given project, he does so honestly. But when he has no reason to either accept or reject a given project, he tries to agree with Mr. Super. If he manages to agree with him he gives himself 10 points; if he is unfavorable when his boss is favorable, he credits himself with 0 points; but when he is favorable and his boss is unfavorable (the worst case), he loses 50 points. The matrix of the game is given in Figure 57. Since the matrix in Figure 57 satisfies condition (ii) of the theorem, it is not strictly determined.

How should one play a nonstrictly determined game? We must first convince ourselves that no single choice is clearly optimal for either player.

Figure 57

		Player C Mr. Super's opinion: Favorable	Unfavorable
Player R Mr. Sub's opinion	Favorable	10	-50
	Unfavorable	0	10

In Example 1, R would like to get one of the 0 payoffs. But if he always chooses \$1 and C finds this out, C can win \$1 by guessing 1. And if R always chooses \$2, then C can win \$2 by guessing 2. Similarly, if C always guesses 1 or always guesses 2, and R finds this out, then R can always get 0. So our first result is that each player must, in some way, prevent the other player from finding out which choice of alternatives he is going to make.

We also note that for a single play of a nonstrictly determined 2×2 game there is no difference between the two strategies, as long as one's strategy is not guessed by the opponent. Let us now consider several plays of the game. What should R do? Clearly, he should not choose the same row all the time, or C will be able to notice and profit by it. Rather, R should choose sometimes one row, sometimes the other. Our key question then is "How often should R choose each of his alternatives?" In Example 1 it seems reasonable that player R (Jones) should choose the \$1 bill about twice as often as the \$2 bill, because his losses, if Smith guesses correctly, are half as much. (We shall see later that this strategy is, indeed, optimal.) In what order should he do this? For instance, should he select the \$1 bill twice in a row and then the \$2 bill? That is dangerous, because if player C (Smith) notices the pattern, he can gain by knowing just what R will do next. Thus we see that R should choose the \$1 bill two-thirds of the time, but according to some unguessable pattern. The only safe way of doing this is to play it two-thirds of the time at random. He could, for instance, roll a die (without letting C see it) and choose the \$1 if 1 through 4 turns up, the \$2 if 5 or 6 turns up. Then his opponent cannot guess what the actual decision will be, since R himself won't know it. We conclude that a rational way of playing is for each player to *mix* his strategies, selecting sometimes one, sometimes the other; and these strategies should be selected at random, according to certain fixed ratios (probabilities) of selecting each.

By a *mixed strategy* in a 2×2 game for player R we shall mean a command of the form "Play row 1 with probability p_1 and play row 2 with probability p_2," where we assume that $p_1 \geq 0$ and $p_2 \geq 0$ and $p_1 + p_2 = 1$. Similarly, a mixed strategy for player C is a command of the form "Play column 1 with probability q_1 and play column 2 with probability q_2," where $q_1 \geq 0$, $q_2 \geq 0$, and $q_1 + q_2 = 1$. A mixed-strategy vector for player R is the probability row vector (p_1, p_2), and a mixed-strategy vector for player C is the probability column vector $\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$.

Examples of mixed strategies are $(\frac{1}{2}, \frac{1}{2})$ and $\begin{pmatrix} \frac{1}{5} \\ \frac{4}{5} \end{pmatrix}$. The reader may wonder how a player could actually play one of these strategies. The mixed strategy $(\frac{1}{2}, \frac{1}{2})$ is easy to realize, since it can be realized by flipping a coin and choosing one alternative if heads turns up and the other alternative if tails turns up. The mixed strategy $\begin{pmatrix} \frac{1}{5} \\ \frac{4}{5} \end{pmatrix}$ is more difficult to realize, since no chance device in common use gives these probabilities. However, suppose a pointer is constructed with a card that is $\frac{4}{5}$ shaded and $\frac{1}{5}$ unshaded, as in Figure 58,

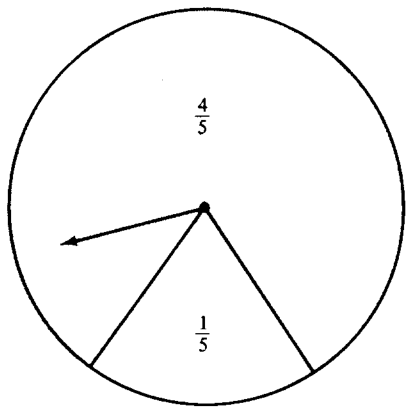


Figure 58

and C simply spins the pointer (without letting R see it, of course!). Then, if the pointer stops on the unshaded part he plays the first column, and if it stops on the shaded part he plays the second column, thus realizing the desired strategy. By varying the proportion of shaded area on the card, other mixed strategies can conveniently be realized. An equally effective and less mechanical device for realizing a given mixed strategy is to use a table of random digits. For the strategy $\begin{pmatrix} \frac{1}{5} \\ \frac{4}{5} \end{pmatrix}$, for example, we could let the digits 0 and 1 represent a play of column 1, and the remaining digits a play of column 2.

We now want to define what we shall mean by a solution to an $m \times n$ matrix game.

Definition Let G be an $m \times n$ matrix with entries g_{ij}. An m-component row vector p is a *mixed-strategy vector* for player R if it is a probability vector; similarly, an n-component column vector q is a *mixed-strategy vector for* C if it is a probability vector. (Recall from Chapter 4 that a probability vector is one with nonnegative entries whose sum is 1.) Let v be a number, let e be an m-component row vector all of whose entries are 1, and let f be an n-component column vector all of whose entries are 1. It follows that the vectors ve and vf are

$$ve = \underbrace{(v, v, \ldots, v)}_{m \text{ components}} \quad \text{and} \quad vf = \left.\begin{pmatrix} v \\ v \\ \vdots \\ v \end{pmatrix}\right\} n \text{ components}$$

Then v is the *value* of the matrix game G and p^0 and q^0 are *optimal strategies* for the players if and only if the following inequalities hold:

(1) $$p^0G \geq ve,$$

(2) $$Gq^0 \leq vf.$$

EXAMPLE 3 In Example 1 of the previous section we had the matrix:

$$G = \begin{pmatrix} 50 & 60 \\ 40 & 50 \end{pmatrix}.$$

We found that the value of this game was $v = 50$ and that optimal strategies were for R to choose row 1, which corresponds to the mixed-strategy vector $p^0 = (1, 0)$, and for C to choose column 1, which corresponds to the mixed-strategy vector $q^0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Carrying out the calculations in (1) and (2), we have

$$p^0G = (1, 0)\begin{pmatrix} 50 & 60 \\ 40 & 50 \end{pmatrix} = (50, 60) \geq (50, 50) = 50(1, 1) = ve$$

and

$$Gq^0 = \begin{pmatrix} 50 & 60 \\ 40 & 50 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 50 \\ 40 \end{pmatrix} \leq \begin{pmatrix} 50 \\ 50 \end{pmatrix} = 50\begin{pmatrix} 1 \\ 1 \end{pmatrix} = vf.$$

In a similar manner the solutions to Examples 2, 3, and 4 of Section 8 can be shown to satisfy the definition above (see Exercises 5, 6, and 7). In Exercise 16 you will be asked to show that optimal strategies to any strictly determined game satisfy the definition above.

Let us return now to the nonstrictly determined 2×2 game. Consider the nonstrictly determined game

$$G = \begin{array}{|c|c|} \hline a & b \\ \hline c & d \\ \hline \end{array}$$

Having argued, as above, that the players should use mixed strategies in playing a nonstrictly determined game, it is still necessary to decide how to choose an optimal mixed strategy.

If R chooses a mixed strategy $p = (p_1, p_2)$ and (independently) C chooses a mixed strategy $q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$, then player R obtains the payoff a with probability p_1q_1; he obtains the payoff b with probability p_1q_2; he obtains c with probability p_2q_1; and he obtains d with probability p_2q_2; hence his mathematical expectation (see Chapter 3, Section 11) is given by the expression

$$ap_1q_1 + bp_1q_2 + cp_2q_1 + dp_2q_2.$$

By a similar computation, one can show that player C's expectation is the negative of this expression.

To justify this definition we must show that if v, p^0, q^0 exist for G, each player can guarantee himself an expectation of v. Let q be any strategy for C. Multiplying (1) on the right by q, we get

$$p^0Gq \geq (v, v)q = v,$$

which shows that, regardless of how C plays, R can assure himself of an expectation of at least v. Similarly, let p be any strategy vector for R. Multiplying (2) on the left by p, we obtain

$$pGq^0 \leq p\begin{pmatrix} v \\ v \end{pmatrix} = v,$$

which shows that, regardless of how R plays, C can assure himself of an expectation of at most v. It is in this sense that p^0 and q^0 are optimal. It follows further that, if both players play optimally, then R's expectation is exactly v and C's expectation is exactly v. Hence we call v the (expected) *value* of the game.

We must now see whether there are strategies p^0 and q^0 for the game G. For complicated games the finding of optimal strategies will be discussed in Section 11. For a 2×2 nonstrictly determined game the following formulas provide the solution:

$$p_1^0 = \frac{d - c}{a + d - b - c}, \tag{3}$$

$$p_2^0 = \frac{a - b}{a + d - b - c}, \tag{4}$$

$$q_1^0 = \frac{d - b}{a + d - b - c}, \tag{5}$$

$$q_2^0 = \frac{a - c}{a + d - b - c}, \tag{6}$$

$$v = \frac{ad - bc}{a + d - b - c}. \tag{7}$$

It is an easy matter to verify (see Exercise 12) that formulas (3)–(7) satisfy conditions (1)–(2). Actually, the inequalities in (1) and (2) become equalities in this simple case, a fact that is not true in general for nonstrictly determined games of larger size.

The denominator in each formula is the difference between the sums of the entries on the two diagonals. Since, for a nonstrictly determined game, the entries on one diagonal must be larger than those on the other, the denominator cannot be zero.

Let us use these formulas to solve the examples mentioned earlier.

EXAMPLE 1 (continued) Applying formulas (3)-(7) to the matrix in Figure 56, we have

$$p_1^0 = \frac{-2-0}{-1-2-0-0} = \frac{2}{3}, \qquad p_2^0 = \frac{-1-0}{-3} = \frac{1}{3},$$

$$q_1^0 = \frac{-2-0}{-3} = \frac{2}{3}, \qquad q_2^0 = \frac{-1-0}{-3} = \frac{1}{3}, \qquad v = \frac{(-1)(-2)-0}{-3} = -\frac{2}{3}.$$

Thus the game is biased in favor of player C, since $v = -\frac{2}{3}$, and optimal strategies are

$$p^0 = (\tfrac{2}{3}, \tfrac{1}{3}) \quad \text{and} \quad q^0 = \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \end{pmatrix}.$$

Both Jones and Smith should select their first alternative two-thirds of the time, according to some random pattern.

EXAMPLE 2 (continued) Let us apply the formulas (3)-(7) to the matrix in Figure 57. We obtain

$$a + d - c - b = 10 + 10 - 0 + 50 = 70,$$

so that:

$$p_1^0 = \frac{10-0}{70} = \frac{1}{7}, \qquad p_2^0 = \frac{10+50}{70} = \frac{6}{7},$$

$$q_1^0 = \frac{10+50}{70} = \frac{6}{7}, \qquad q_2^0 = \frac{10-0}{70} = \frac{1}{7}, \qquad v = \frac{10 \cdot 10 - 0}{70} = \frac{10}{7}.$$

Notice that the game is biased in favor of Mr. Sub, not his boss Mr. Super! Also Mr. Sub's optimal strategy is to have an *unfavorable* opinion 6 out of 7 times, while Mr. Super's optimal strategy is to have a *favorable* opinion 6 out of 7 times! Thus, if this game is at all realistic, a subordinate should be much more critical than his superior when judging situations in which there is no clear-cut reason to either accept or reject a project. The conclusion is based on game-theory analysis, not on the two persons' relative ages, experience, and so on.

We conclude this section by proving three theorems that characterize the value and optimal strategies of a game.

Theorem If G is a matrix game that has a value and optimal strategies, then the value of the game is unique.

Proof Suppose that v and w are two different values for the game G. Then let p^0 and q^0 be optimal mixed-strategy vectors associated with the value v such that

$$\text{(a)} \qquad p^0G \geq ve,$$
$$\text{(b)} \qquad Gq^0 \leq vf.$$

Similarly, let p^1 and q^1 be optimal mixed-strategy vectors associated with the value w such that

$$\text{(c)} \qquad p^1G \geq we,$$
$$\text{(d)} \qquad Gq^1 \leq wf.$$

If we now multiply (a) on the right by q^1, we get $p^0Gq^1 \geq (ve)q^1 = v$. In the same way, multiplying (d) on the left by p^0 gives $p^0Gq^1 \leq w$. The two inequalities just obtained show that $w \geq v$.

Next we multiply (b) on the left by p^1 and (c) on the right by q^0, obtaining $v \geq p^1Gq^0$ and $p^1Gq^0 \geq w$, which together imply that $v \geq w$.

Finally we see that $v \leq w$ and $v \geq w$ imply together that $v = w$—that is, the value of the game is unique.

Theorem If G is a matrix game with value v and optimal strategies p^0 and q^0, then $v = p^0Gq^0$.

Proof By definition v, p^0, and q^0 satisfy

$$p^0G \geq ve \quad \text{and} \quad Gq^0 \leq vf.$$

Multiplying the first of these inequalities on the right by q^0, we get $p^0Gq^0 \geq v$. Similarly, multiplying the second inequality on the left by p^0, we obtain $p^0Gq^0 \leq v$. These two inequalities together imply that $v = p^0Gq^0$, concluding the proof.

The theorems just proved are important because they permit us to interpret the *value* of a game as an *expected value* (see Chapter 3, Section 11). Briefly the interpretation is the following: If the game G is played repeatedly and if each time it is played player R uses the mixed strategy p^0 and player C uses the mixed strategy q^0, then the value v of G is the expected value of the game for R. The law of large numbers implies that, if the number *of plays of* G is sufficiently large, then the average value of R's winnings will (with high probability) be arbitrarily close to the value v of the game G.

As an example, let G be the matrix of the game of matching pennies:

$$G = \begin{array}{|c|c|} \hline 1 & -1 \\ \hline -1 & 1 \\ \hline \end{array}$$

Using the formulas above, we find that optimal strategies in this game are for R to choose each row with probability $\frac{1}{2}$ and for C to choose each column with probability $\frac{1}{2}$. The value of G is zero. Notice that the only two payoffs that result from a single play of the game are $+1$ and -1, neither of which is equal to the value of the game. However, if the game is played repeatedly, the average value of R's payoffs will approach zero, which is the value of the game.

Theorem If G is a game with value v and optimal strategies p^0 and q^0, then v is the largest expectation that R can assure for himself. Similarly, v is the smallest expectation that C can assure for himself.

Proof Let p be any mixed-strategy vector of R and let q^0 be an optimal strategy for C; then multiply the equation $Gq^0 \leq vf$ on the left by p, obtaining $pGq^0 \leq v$. The latter equation shows that, if C plays optimally, the most that R can assure for himself is v. Now let p^0 be optimal for R; then, for every $q, p^0Gq \geq v$, so that R can actually assure himself of an expection of v. The proof of the other statement of the theorem is similar.

The theorem above gives an intuitive justification to the definition of value and optimal strategies for a game. Thus the value is the "best" that a player can assure himself, and optimal strategies are the means of assuring this "best."

EXERCISES

1. Find the optimal strategies for each player and the values of the following games:

(a)

1	2
3	4

(b)

1	0
−1	2

(c)

2	3
1	4

(d)

15	3
−1	2

(e)

7	−6
5	8

(f)

3	15
−1	10

[*Ans.* (a) $v = 3$; $(0, 1)$; $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$; (b) $v = \frac{1}{2}$; $(\frac{3}{4}, \frac{1}{4})$; $\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$;

(d) $v = 3$; $(1, 0)$; $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$; (e) $v = \frac{43}{8}$; $(\frac{3}{16}, \frac{13}{16})$; $\begin{pmatrix} \frac{7}{8} \\ \frac{1}{8} \end{pmatrix}$.]

2. Set up the ordinary game of matching pennies as a matrix game. Find its value and optimal strategies. How are the optimal strategies realized in practice by players of this game?

3. A version of two-finger Morra is played as follows: Each player holds up either one or two fingers; if the sum of the number of fingers shown is even, player R gets the sum, and if the sum is odd, player C gets it.
   (a) Show that the game matrix is

$$\begin{array}{cc|c|c|} & & \multicolumn{2}{c}{\text{Player C}} \\ & & 1 & 2 \\ \hline \text{Player R} & 1 & 2 & -3 \\ \hline & 2 & -3 & 4 \\ \hline \end{array}$$

   (b) Find optimal strategies for each player and the value of the game.

   [*Ans.* $(\frac{7}{12}, \frac{5}{12})$, $\begin{pmatrix} \frac{7}{12} \\ \frac{5}{12} \end{pmatrix}$, $v = -\frac{1}{12}$.]

4. Rework Exercise 3 if player C gets the even sum and player R gets the odd sum.
5. Let G be the matrix in Figure 51 described in Example 2 of Section 8. With $v = 0$, $p^0 = (0, 1)$, and $q^0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, show that formulas (1) and (2) are satisfied.
6. Show that the strategies derived in Example 3 of Section 8 satisfy formulas (1) and (2).
7. Show that the strategies derived in Example 4 of Section 8 satisfy formulas (1) and (2).
8. If

$$G = \begin{array}{|c|c|} \hline a & b \\ \hline c & d \\ \hline \end{array}$$

   is nonstrictly determined, prove that it is fair if and only if $ad = bc$.
9. In formulas (3)–(7) prove that $p_1 > 0, p_2 > 0, q_1 > 0$, and $q_2 > 0$. Must v be greater than zero?
10. Find necessary and sufficient conditions that the game

$$G = \begin{array}{|c|c|} \hline a & 0 \\ \hline 0 & b \\ \hline \end{array}$$

   be nonstrictly determined. Find optimal strategies for each player and the value of G, if it is nonstrictly determined.
   [*Ans.* a and b must be both positive or both negative. $p_1 = b/(a + b)$; $p_2 = a/(a + b)$; $q_1 = b/(a + b)$; $q_2 = a/(a + b)$; $v = ab/(a + b)$.]

11. Suppose that player R tries to find C in one of three towns X, Y, and Z. The distance between X and Y is five miles, the distance between Y and Z is five miles, and the distance between Z and X is ten miles. Assume that R and C can each go to one and only one of the three towns and that if they both go to the same town R "catches" C; otherwise C "escapes." Credit R with ten points if he catches C, and credit C with a number of points equal to the number of miles he is away from R if he escapes.
    (a) Set up the game matrix.
    (b) Show that both players have the same optimal strategy, namely, to go to towns X and Z with equal probabilities and to go to town Y with probability $\frac{1}{4}$.
    (c) Find the value of the game.
12. Verify that formulas (3)–(7) satisfy conditions (1) and (2).
13. Consider the (symmetric) game whose matrix is

$$G = \begin{array}{|c|c|c|} \hline 0 & -a & -b \\ \hline a & 0 & -c \\ \hline b & c & 0 \\ \hline \end{array}$$

    (a) If a and b are both positive or both negative, show that G is strictly determined.
    (b) If b and c are both positive or both negative, show that G is strictly determined.
    (c) If $a > 0$, $b < 0$, and $c > 0$, show that an optimal strategy for player R is given by

$$\left(\frac{c}{a-b+c}, \quad \frac{-b}{a-b+c}, \quad \frac{a}{a-b+c}\right).$$

    (d) In part (c) find an optimal strategy for player C.
    (e) If $a < 0$, $b > 0$, and $c < 0$, show that the strategy given in (c) is optimal for R. What is an optimal strategy for player C?
    (f) Prove that the value of the game is always zero.
14. In a well-known children's game each player says "stone" or "scissors" or "paper." If one says "stone" and the other "scissors," then the former wins a penny. Similarly, "scissors" beats "paper," and "paper" beats "stone." If the two players name the same item, then the game is a tie.
    (a) Set up the game matrix.
    (b) Use the results of Exercise 13 to solve the game.
15. In Exercise 14 let us suppose that the payments are different in different cases. Suppose that when "stone breaks scissors" the payment is one cent; when "scissors cut paper" the payment is two cents; and when "paper covers stone" the payment is three cents.

(a) Set up the game matrix.
(b) Use the results of Exercise 13 to solve the game.
[*Ans.* $\frac{1}{3}$ "stone," $\frac{1}{2}$ "scissors," $\frac{1}{6}$ "paper"; $v = 0$.]

16. A strictly determined $m \times n$ matrix game G contains a saddle entry g_{ij} that is simultaneously the minimum of row i and the maximum of column j.
(a) Show that by rearranging rows and columns (if necessary) we can assume that g_{11} is a saddle value.
(b) Let $v = g_{11}$ and p^0 and q^0 be probability vectors with first component equal to 1 and all other components equal to 0. Show that these quantities satisfy (1) and (2).

17. Verify that the strategies $p^0 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and

$$q^0 = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}$$

are optimal in the game G whose matrix is

$$G = \begin{array}{|c|c|c|} \hline 1 & 0 & 0 \\ \hline 0 & 1 & 0 \\ \hline 0 & 0 & 1 \\ \hline \end{array}$$

What is the value of the game?

18. Generalize the result of Exercise 16 to the game G whose matrix is the $n \times n$ identity matrix.

19. Consider the following game:

$$G = \begin{array}{|c|c|c|} \hline a & 0 & 0 \\ \hline 0 & b & 0 \\ \hline 0 & 0 & c \\ \hline \end{array}$$

(a) If a, b, and c are not all of the same sign, show that the game is strictly determined with value zero.
(b) If a, b, and c are all of the same sign, show that the vector

$$\left(\frac{bc}{ab + bc + ca}, \frac{ca}{ab + bc + ca}, \frac{ab}{ab + bc + ca}\right)$$

is an optimal strategy for player R.
(c) Find player C's optimal strategy for case b.
(d) Find the value of the game for case b, and show that it is positive if a, b, and c are all positive, and negative if they are all negative.

20. Suppose that the entries of a matrix game are rewritten in new units (e.g., dollars instead of cents). Show that the monetary value of the game has not changed.
21. Consider the game of matching pennies whose matrix is

1	−1
−1	1

If the entries of the matrix represent gains or losses of one penny, would you be willing to play the game at least once? If the entries represent gains or losses of one dollar, would you be willing to play the game at least once? If they represent gains or losses of one million dollars, would you play the game at least once? In each of these cases show that the value is zero and optimal strategies are the same. Discuss the practical application of the theory of games in the light of this example.

10 SOLVING MATRIX GAMES BY A GEOMETRIC METHOD

In Section 8 we found that a strictly determined game of any size could be solved almost by inspection. In Section 9 we found formulas for solving nonstrictly determined 2×2 games. In Section 11 we shall discuss the application of the simplex method to solve arbitrary $m \times n$ matrix games. In the present section we shall discuss special matrix games in which one of the players has just two strategies, and we shall find that a simple geometric method suffices to solve such games rather easily.

EXAMPLE 1 Suppose that Jones conceals one of the following four bills in his hand: a \$1 or a \$2 United States bill or a \$1 or a \$2 Canadian bill. Smith guesses either "United States" or "Canadian" and gets the bill if his guess is correct. We assume that a Canadian dollar has the same real value as a United States dollar. The matrix of the game is the following:

			Smith guesses: U.S.	Smith guesses: Can.
Jones chooses	U.S.	\$1	−1	0
		\$2	−2	0
	Can.	\$1	0	−1
		\$2	0	−2

It is obvious that Jones should always choose the \$1 bill of either country rather than the \$2 bill, since by doing so he may cut his losses and will never increase them. This can be observed in the matrix above, since every entry in the second row is less than or equal to the corresponding entry in the first row, and every entry in the fourth row is less than or equal to the corresponding entry in the third row. In effect we can eliminate the second and fourth rows and reduce the game to the following 2×2 matrix game:

		Smith guesses U.S.	Smith guesses Can.
Jones chooses	U.S. \$1	-1	0
	Can. \$1	0	-1

The new matrix game is nonstrictly determined with optimal strategies $(\frac{1}{2}, \frac{1}{2})$ for Jones and $\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$ for Smith. The value of the game is $-\frac{1}{2}$, which means that Smith should be willing to pay 50 cents to play it.

Definition Let A be an $m \times n$ matrix game. We shall say that row i *dominates* row h if every entry in row i is as large as or larger than the corresponding entry in row h. Similarly, we shall say that column j *dominates* column k if every entry in column j is as small as or smaller than the corresponding entry in column k.

Any dominated row or column can be omitted from the matrix game without materially affecting its solution. In the original matrix of Example 1 above, we see that row 1 dominates row 2, and also that row 3 dominates row 4.

EXAMPLE 2 Consider the game whose matrix is:

$$G = \begin{bmatrix} 1 & 0 & -1 & 0 \\ -3 & -2 & 1 & 2 \end{bmatrix}$$

Observe that column 2 and column 3 each dominate column 4; that is, player C should never play the last column. Thus the game can be reduced to the following 2×3 game:

$$G' = \begin{bmatrix} 1 & 0 & -1 \\ -3 & -2 & 1 \end{bmatrix}$$

No further rows or columns can be omitted because of domination; hence we must introduce a new technique for the solution of this game.

Suppose that player R announces he is going to use the mixed strategy $p = (p_1, p_2)$. Using the relation $p_1 = 1 - p_2$, we can write this as $p = (1 - p_2, p_2)$. Assume for the moment that player C knows R will use this strategy. Then he can compute his expected payment y from choosing each of his alternatives in G' as follows:

If he chooses column 1:

$$y = 1 \cdot p_1 - 3 \cdot p_2 = (1 - p_2) - 3p_2 = 1 - 4p_2.$$

If he chooses column 2:

$$y = 0 \cdot p_1 - 2 \cdot p_2 = -2p_2.$$

If he chooses column 3:

$$y = -1 \cdot p_1 + 1 \cdot p_2 = -(1 - p_2) + p_2 = -1 + 2p_2.$$

Notice that each of these expectations expresses y as a linear function of p_2. Hence the graphs of these expectations will be a straight line in each case. Since we have the restriction $0 \leq p_2 \leq 1$, we are interested only in the part of the line for which p_2 satisfies the restriction. In Figure 59 we

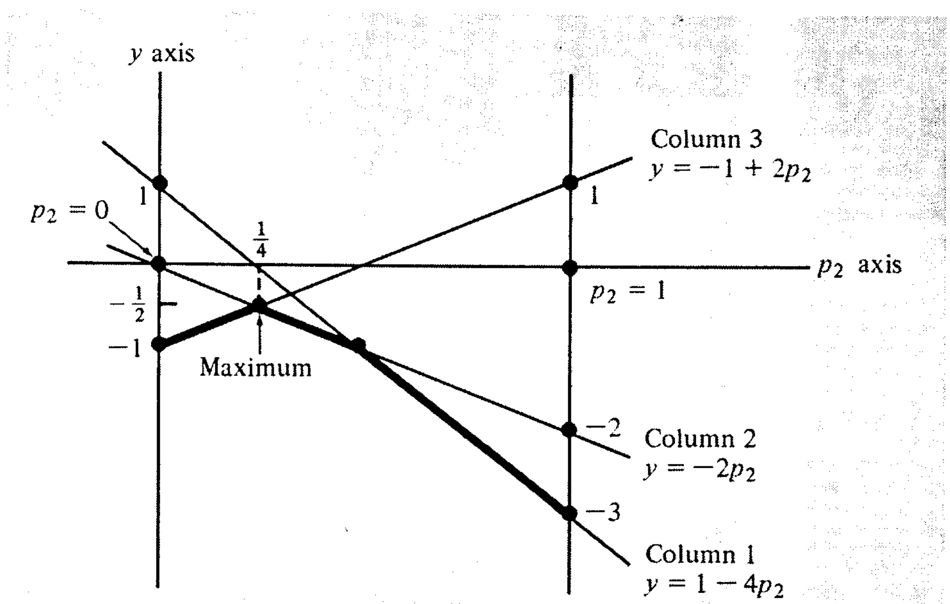


Figure 59

have shown p_2 plotted on the horizontal axis and y on the vertical axis. We have also drawn the vertical line at $p_2 = 1$. The graphs of each of the lines above are shown. Observe that the ordinates of each line when $p_2 = 0$ are just the entries in the first row of G', and the ordinates of each line when $p_2 = 1$ are just the entries in the second row. Since we can easily find these two distinct points on each line, it is easy to draw them.

We now can analyze what C will do. For each value of p_2 that completely determines R's mixed strategy $p = (1 - p_2, p_2)$, player C will minimize his own expectation—that is, he will choose the lowest of the three lines plotted in Figure 59. For each p_2 the lowest line has been drawn in heavily, resulting in the broken-line function shown in the figure. Now R is the maximizing player, so he will try to get the maximum of this function. By visual inspection this obviously occurs at the intersection of the lines corresponding to column 2 and column 3, when $p_2 = \frac{1}{4}$ and the "height" of this function at that point is $-\frac{1}{2}$. From the figure it is clear that $-\frac{1}{2}$ is the maximum R can assure himself, and he can obtain this by using the strategy $p = (\frac{3}{4}, \frac{1}{4})$ corresponding to $p_2 = \frac{1}{4}$. We can find optimal strategies for player C by considering the 2×2 subgame of G (and G') consisting of the second and third columns:

$G'' =$

0	−1
−2	1

Applying the formulas of the preceding section, we obtain as optimal strategies:

$$p^0 = (\tfrac{3}{4}, \tfrac{1}{4}), \qquad q^0 = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}, \qquad v = -\tfrac{1}{2}.$$

We can extend q^0 to an optimal strategy for player C in G by adding two zero entries thus:

$$q^0 = \begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}.$$

Player R's strategy and the value remain the same, as the reader can easily verify.

EXAMPLE 3 We have already seen examples where a player has more than one optimal strategy. The game whose matrix is

$G =$

3	1	0
0	1	3

is another example. To carry out the same kind of analysis as before, assume that R chooses $p = (p_1, p_2) = (1 - p_2, p_2)$. Then

If C chooses column 1: $y = 3(1 - p_2) = 3 - 3p_2$.
If C chooses column 2: $y = (1 - p_2) + p_2 = 1$.
If C chooses column 3: $y = 3p_2$.

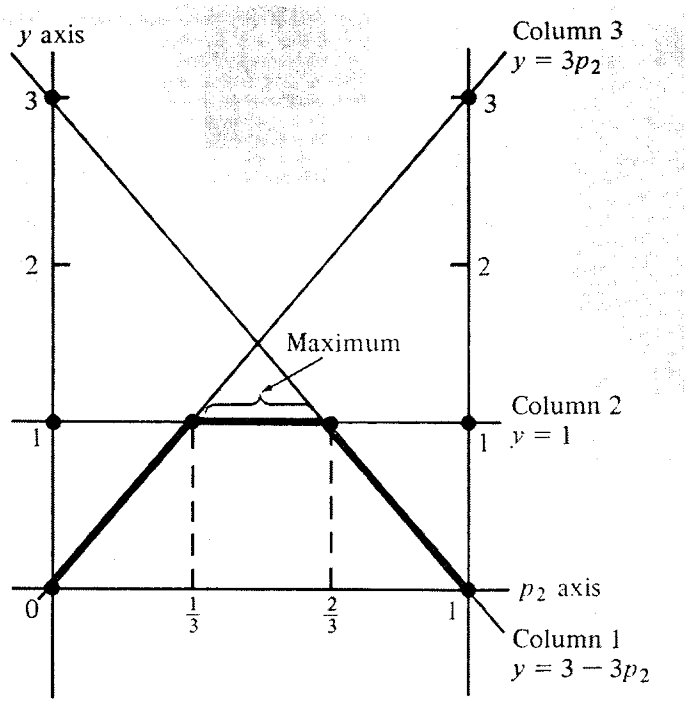


Figure 60

The graphs of these three functions are shown in Figure 60, and the minimum of the three is shown darkened. Since the darkened graph has a flat area on the top, the entire flat area represents the maximum of the function. The endpoints of the flat area are $(\frac{2}{3}, \frac{1}{3})$ and $(\frac{1}{3}, \frac{2}{3})$, and the intervening points that are convex combinations of these, such as

$$a(\tfrac{2}{3}, \tfrac{1}{3}) + (1 - a)(\tfrac{1}{3}, \tfrac{2}{3}) = \tfrac{1}{3}(a + 1, 2 - a),$$

are also optimal strategies, as the reader can verify by inspection. The unique optimal strategy for the column player is to choose the second column, so $q^0 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. Of course, $v = 1$.

Theorem The set of optimal strategies for either player in a matrix game is a convex set. That is, if p^0 and r^0 are optimal for player R, then $ap^0 + (1 - a)r^0$ is also optimal for him, for any a in the range $0 \leq a \leq 1$. Similarly, if q^0 and s^0 are optimal for player C, then so is $aq^0 + (1 - a)s^0$ for a in the same range.

We shall not give a formal proof of the theorem here, but it is clearly illustrated in Figure 60. In the next section we shall show that a matrix game is equivalent to a linear programming problem, and then the theorem becomes a consequence of the corresponding theorem in linear programming.

EXAMPLE 4 So far we have illustrated cases in which the row player had just two strategies and the column player had three or more. A similar method works to solve games in which the column player has just two strategies and the row player has more. Consider the game whose matrix is

$$G = \begin{array}{|c|c|} \hline 6 & -1 \\ \hline 0 & 4 \\ \hline 4 & 3 \\ \hline \end{array}$$

Suppose we reverse the analysis above and assume that the column player selects a mixed strategy

$$q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} 1 - q_2 \\ q_2 \end{pmatrix}$$

and then considers what action R will take. Again there are three choices:

If he chooses row 1: $y = 6q_1 - q_2 = 6(1 - q_2) - q_2 = 6 - 7q_2$.
If he chooses row 2: $y = 4q_2$.
If he chooses row 3: $y = 4q_1 + 3q_2 = 4(1 - q_2) + 3q_2 = 4 - q_2$.

In each case y is the expectation that player R has for each choice. Since he is the maximizing player, he will want to maximize his expectation. In Figure 61 we have shown the three straight lines corresponding to each of

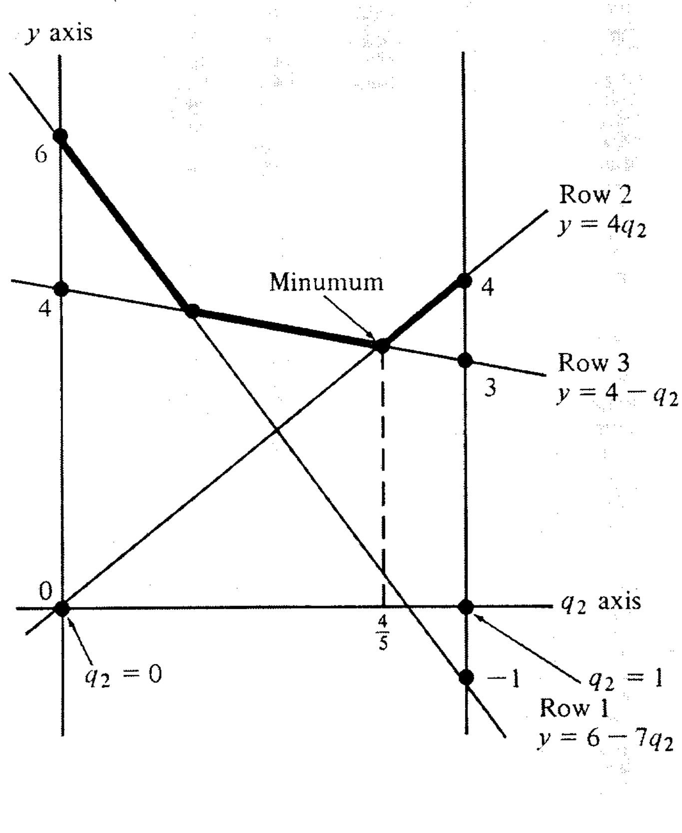


Figure 61

these expectations and have darkened the *maximum* of each of these. Player C will want to choose the smallest value on the darkened broken-line function marked in the figure. Since it corresponds to $p_2 = \frac{4}{5}$, the corresponding optimal strategy for the column player is $\begin{pmatrix} \frac{1}{5} \\ \frac{4}{5} \end{pmatrix}$.

To find the corresponding optimal strategy for the row player we consider the 2×2 in the last two rows of the matrix:

0	4
4	3

Using the formulas of the previous section, we have optimal strategies:

$$p^0 = (\tfrac{1}{5}, \tfrac{4}{5}), \qquad q^0 = \begin{pmatrix} \frac{1}{5} \\ \frac{4}{5} \end{pmatrix}, \qquad v = \tfrac{16}{5}.$$

We can extend the optimal row strategy to one optimal for the original game by adding a zero. Thus

$$p^0 = (0, \tfrac{1}{5}, \tfrac{4}{5})$$

is optimal in the game G originally stated.

By using graph paper and a ruler, the reader will be able to solve in a similar manner other games in which one of the players has just two strategies. In principle the graphical method could be extended to larger games, but it is difficult to draw three-dimensional graphs and impossible to draw four- and higher-dimensional graphs, so that this idea has limited usefulness.

The geometric ideas presented in this section are useful conceptually. For instance, the following theorem is intuitively obvious from the geometric point of view.

Theorem Let G be an $m \times n$ matrix game with value v; let E be the $m \times n$ matrix each of whose entries is 1; and let k be *any* constant. Then the game $G + kE$ has value $v + k$, and every strategy optimal in the game G is also optimal in the game $G + kE$. (Note that the game $G + kE$ is obtained from the game G by adding the number k to each entry in G.)

If we apply this theorem to any of the previous examples, its truth is clear, since adding k to each entry in G merely moves all the lines in each graph up or down by the same amount. Hence the locations of the optimum points are unchanged, and the value is changed by the amount k.

One consequence of this theorem is the fact that a matrix game G can be replaced by an equivalent game all of whose entries are positive and whose value is positive. One simply chooses a sufficiently large k and forms the game $G + kE$. We shall use this fact in the next section.

EXERCISES

1. Solve the following games:

(a)

3	0
−2	3
7	5

[*Ans.* $v = 5$; $(0, 0, 1)$; $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.]

(b)

10	5	4	6
18	3	3	4

(c)

1	0	2
0	3	2

[*Ans.* $v = \frac{3}{4}$; $(\frac{3}{4}, \frac{1}{4})$; $\begin{pmatrix} \frac{3}{4} \\ \frac{1}{4} \\ 0 \end{pmatrix}$.]

(d)

0	2
1	3
−1	0
2	0

(e)

1	2	3
4	2	1

[*An ans.* $v = 2$; $(\frac{3}{5}, \frac{2}{5})$; $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.]

(f)

1	0	1	1	2
0	−1	−2	−3	−10

2. Solve the following games:

(a)

0	15
8	0
−10	20
10	12

(b)

−1	−2	0	−3	−4
−2	1	0	2	5

(c)

-1	5	-1	-2	8	10
3	-6	0	8	-9	-8

[*An ans.* $v = -\frac{1}{2}$; $(\frac{1}{2}, \frac{1}{2})$; $\begin{pmatrix} 0 \\ \frac{1}{12} \\ \frac{11}{12} \\ 0 \\ 0 \\ 0 \end{pmatrix}$.]

3. Solve the game

1	2	3
3	2	1

Since there is more than one optimal strategy for C, find a range of optimal strategies for him.

4. Consider the game whose matrix is

13	-7
3	8
-1	14
9	-1

(a) Find player C's optimal strategy by graphical means.
(b) Show that there are six possible subgames that can be chosen by player R.
(c) Of the six possible subgames show that two are strictly determined and do not give optimal strategies in the original game.
(d) Show that the other four subgames have solutions that can be extended to optimal strategies in the original game.

[*Ans.* $(\frac{1}{5}, \frac{4}{5}, 0, 0)$, $(\frac{3}{7}, 0, \frac{4}{7}, 0)$, $(0, 0, \frac{2}{5}, \frac{3}{5})$, $(0, \frac{2}{3}, 0, \frac{1}{3})$.]

5. Suppose that Jones conceals in his hand one, two, three, or four silver dollars and Smith guesses "even" or "odd." If Smith's guess is correct, he wins the amount that Jones holds; otherwise he must pay Jones this amount. Set up the corresponding matrix game and find an optimal strategy for each player in which he puts positive weight on all his (pure) strategies. Is the game fair?

6. Consider the following game: Player R announces "one" or "two"; then, independently of each other, both players write down one of these two numbers. If the sum of the three numbers so obtained is odd,

C pays R the odd sum in dollars; if the sum of the three numbers is even, R pays C the even sum in dollars.

(a) What are the strategies of R? [*Hint:* He has four strategies.]

(b) What are the strategies of C? [*Hint:* We must consider what C does after "one" is announced or after a "two." Hence he has four strategies.]

(c) Write down the matrix for the game.

(d) Restrict player R to announcing "two," and allow for C only those strategies where his number does not depend on the announced number. Solve the resulting 2×2 game.

(e) Extend the above mixed strategies to the original game, and show that they are optimal.

(f) Is the game favorable to R? If so, by how much?

7. Answer the same questions as in Exercise 6 if R gets the even sum and C gets the odd sum [except that, in part (d), restrict R to announce "one"]. Which game is more favorable for R? Could you have predicted this without the use of game theory?

8. Two players play five-finger Morra by extending from one to five fingers: If the sum of the number of fingers is even, R gets one, while if the sum is odd, C gets one. Suppose that each player shows only one or two fingers. Show that the resulting game is like matching pennies. Show that the optimal strategies for this game, when extended, are optimal in the whole game.

9. A version of three-finger Morra is played as follows: Each player shows from one to three fingers; R always pays C an amount equal to the number of fingers that C shows; if C shows exactly one more or two fewer fingers than R, then C pays R a positive amount x (where x is independent of the number of fingers shown).

(a) Set up the game matrix for arbitrary x's.

(b) If $x = \frac{1}{2}$, show that the game is strictly determined. Find the value. [*Ans.* $v = -\frac{5}{2}$.]

(c) If $x = 2$, show that there is a pair of optimal strategies in which the first player shows one or two fingers and the second player shows two or three fingers. [*Hint:* Use domination.] Find the value. [*Ans.* $v = -\frac{3}{2}$.]

(d) If $x = 6$, show that an optimal strategy for R is to use the mixed strategy, $(\frac{1}{3}, \frac{1}{2}, \frac{1}{6})$. Show that the optimal mixed strategy for C is to choose his three strategies each with probability $\frac{1}{3}$. Find the value of the game.

10. Another version of three-finger Morra goes as follows: Each player shows from one to three fingers; if the sum of the number of fingers is even, then R gets an amount equal to the number of fingers that C shows: if the sum is odd, C gets an amount equal to the number of fingers that R shows.

(a) Set up the game matrix.

(b) Reduce the game to a 2×2 matrix game.

(c) Find optimal strategies for each player and show that the game is fair.

11. Consider the game:

a	b
c	d

(a) Draw the graph of expectations for the row player when $a = b$ and prove graphically that the game is strictly determined.
(b) Draw the same graph when $a > b$, $a > c$, $d > b$, $d > c$, and show that the game is nonstrictly determined.
(c) Draw the same graph when $a < b$, $a < c$, $d < b$, $d < c$, and show that the game is nonstrictly determined.
(d) Draw graphs to illustrate cases in which (b) and (c) do not hold and show that the resulting game is strictly determined.

12. Consider the game of Exercise 11 with the same amount k added to each matrix entry. Show graphically that the value of the game is changed by k and that optimal strategies are the same in both games.

The remaining exercises refer to the *product payoff game* (due to A. W. Tucker). Two sets, S and T, are given, each set containing at least one positive and at least one negative number (but no zeros). Player R selects a number s from set S, and player C selects a number t from set T. The payoff is st.

13. Set up the game for the sets $S = \{1, -1, 2\}$ and $T = \{1, -3, 2, -4\}$.

[*Ans.*

1	−3	2	−4
−1	3	−2	4
2	−6	4	−8

.]

14. Consider the following mixed strategy for either player: "Choose a positive number p and a negative number n with probabilities $-n/(p - n)$ and $p/(p - n)$, respectively." Assume that R uses this strategy.
(a) If C chooses a positive number, show that the expected payoff to R is 0.
(b) If C chooses a negative number, show that the expected payoff to R is 0.

15. Rework Exercise 14 with R and C interchanged.

16. Use the results of Exercises 14 and 15 to show that the game is fair, and that the strategy quoted in Exercise 14 is optimal for either player.

17. Find all strategies of the kind indicated in Exercise 14 for both players for the game of Exercise 13.

[*Partial ans.* For R they are $(\frac{1}{2}, \frac{1}{2}, 0)$ and $(0, \frac{2}{3}, \frac{1}{3})$.]

18. By subtracting 10 from each entry, show that the following game is derived from a product payoff game, and find all strategies like those in Exercise 14 for both players. What is the value of the game?

11	7	12	6
9	13	8	14
12	4	14	2

[*Hint:* Use Exercises 13 and 17 and the last theorem.]

19. If a player in the product payoff game has m positive and n negative numbers in his set, show that he has mn strategies like those in Exercise 14.

11 THE SIMPLEX METHOD FOR SOLVING MATRIX GAMES

We have so far restricted our attention to examples of matrix games that were simple enough to be solved by unformalized computations. However, games of realistic size frequently lead to very large matrices for which these simple techniques are not adequate. A clue to a general technique may be found in the fact that the row player is a maximizing player while the column player is a minimizing player. Hence the problems they are trying to solve sound somewhat like the dual linear programming problems of Section 4. So if a matrix game can be formulated as a linear programming problem, it can be solved by the simplex method discussed in Section 5.

There are several ways of showing that a matrix game is equivalent to a linear program. We choose a very simple approach here, based on the fact, stated in Section 10, that every matrix game is equivalent to one in which all entries are positive and hence whose value is positive.

Besides finding an equivalent linear programming problem, we shall give a proof, based on the duality theorem of linear programming, that every matrix game has a solution. And we shall present a simplex format suitable for the solution of any matrix game.

Let G be an $m \times n$ matrix game and let E be the $m \times n$ matrix all of whose entries are 1's. The second theorem of Section 10 states that G and $G + kE$ have the same optimal strategies, and the value of the second game is k plus the value of the first game. Hence if we start with any game G we can replace it by a game G' all of whose entries are positive, and which has the same optimal strategies. For instance, to get G' we could add 1 minus the most negative entry in G to every entry in G.

Thus without loss of generality we let G be a positive matrix game. We also let e be the n-component row vector all of whose entries are 1's, and let f be the m-component column vector all of whose entries are 1's. Let z be an m-component row vector and x an n-component column vector.

We now consider the following dual linear programming problems:

$$(1) \qquad \left\{\begin{array}{l}\text{Minimize } zf \\ \text{subject to:} \\ \quad zG \geq e \\ \quad z \geq 0\end{array}\right. \qquad \left\{\begin{array}{l}\text{Maximize } ex \\ \text{subject to:} \\ \quad Gx \leq f \\ \quad x \geq 0.\end{array}\right.$$

Note that $x = 0$ satisfies the constraints of the maximizing problem; also, because the entries of G and f are positive, there is at least one nonzero x vector that will satisfy these constraints. Moreover, the set of feasible x vectors is bounded. Because of these facts, and because e has all positive entries, the maximizing problem has solution x^0 such that $ex^0 > 0$. Hence, by the duality theorem, the minimum problem has a solution x^0 and

$$t = z^0Gx^0 = z^0f = ex^0 > 0.$$

We now set

$$p^0 = \frac{z^0}{t}, \qquad q^0 = \frac{x^0}{t}, \quad \text{and} \quad v = \frac{1}{t},$$

and observe that p^0 and q^0 are probability vectors.

It is easy to see that p^0 is an optimal strategy for player R in G, since x^0 satisfies the constraints of the minimizing problem, and hence

$$p^0G = \frac{z^0G}{t} \geq \frac{e}{t} = ve.$$

In Exercise 1 the reader is asked to show similarly that $Gq^0 \leq vf$.

We summarize these results in the following theorem:

Theorem (a) Solving the matrix game G with positive entries can be accomplished by solving the dual linear programming problems (1).

(b) Every matrix game has at least one solution; solutions to such games can be found by the simplex method.

Actually, it is not necessary that the matrix game be positive in order that the simplex method work. It is enough that its value be positive. However, in Exercise 3 the reader is asked to work a specific example for which the linear programming problem as described above has no solution if applied to a game with zero value.

Before proceeding to specific examples, let us outline the procedure to be followed in setting up a matrix game for solution by the simplex method.

1. Set up the matrix of the game.
2. Check to see whether the game is strictly determined; if so, the solution is already obtained.
3. Check for row and column dominance and remove dominated rows and columns.

4. Make certain that the value of the game is positive. It is sufficient for this to add 1 minus the most negative matrix entry to every entry of the matrix. Let k be the amount added, if any, to each matrix entry.
5. Let G be the matrix of the resulting game; suppose it is $m \times n$. Construct the following matrix tableau:

G	f
e	0

6. Carry out the steps of the simplex algorithm until all indicators are nonpositive. Determine the solutions z^0 and x^0 to the dual linear programming problems, and let $t = z^0 f = ex^0$. We know $t > 0$.
7. The solutions to the original matrix game are given by

$$p^0 = \frac{z^0}{t}, \qquad q^0 = \frac{x^0}{t} \quad \text{and} \quad v = \frac{1}{t} - k.$$

(If dominated rows or columns were removed from the game, the strategy vectors may have to be extended by the addition of some zero components.)

EXAMPLE 1 We know that the matching-pennies game is fair—that is, it has value zero. To make its value positive, we add $k = 2$ to each matrix entry, yielding the following game:

3	1
1	3

Obviously this game is not strictly determined and it does not have dominated rows or columns. We set up the simplex tableau and solve it as shown in Figure 62(a)–(c). (Note that we have called the variables on the left z_1 and z_2 instead of v_1 and v_2, since we are now using v for the value of the game.) From the final tableau in Figure 62(c) we can see that the value of the game is 2 (the reciprocal of $t = \frac{1}{2}$), so that the value of the matching-pennies game is 0, which we know already. Also, optimal strategies are

$$p^0 = \frac{z^0}{t} = (\tfrac{1}{2}, \tfrac{1}{2}) \quad \text{and} \quad q^0 = \frac{x^0}{t} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix},$$

which we had discovered earlier.

EXAMPLE 2 Let us solve the two-finger Morra game of Exercise 3 of Section 9. To convert the game into one with positive value let us add 3 to each entry of the matrix,

$$\begin{array}{c|cc|c|l}
 & x_1 & x_2 & -1 & \\
\hline
z_1 & 3 & 1 & 1 & = -y_1 \\
z_2 & 1 & 3 & 1 & = -y_2 \\
\hline
-1 & 1 & 1 & 0 & = f \\
\hline
 & = u_1 & = u_2 & = g &
\end{array}$$

(a)

$$\begin{array}{c|cc|c|l}
 & y_1 & x_2 & -1 & \\
\hline
u_1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & = -x_1 \\
z_2 & -\frac{1}{3} & \frac{8}{3} & \frac{2}{3} & = -y_2 \\
\hline
-1 & -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} & = f \\
\hline
 & = z_1 & = u_2 & = g &
\end{array}$$

(b)

$$\begin{array}{c|cc|c|l}
 & y_1 & y_2 & -1 & \\
\hline
u_1 & \frac{3}{8} & -\frac{1}{3} & \frac{1}{4} & = -x_1 \\
u_2 & -\frac{1}{3} & \frac{3}{8} & \frac{1}{4} & = -x_2 \\
\hline
-1 & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{2} & = f \\
\hline
 & = z_1 & = z_2 & = g &
\end{array}$$

Figure 62 (c)

noting that this will give two zeros in the resulting game matrix. These zeros will simplify the simplex calculations. The game matrix now is

5	0
0	7

Figure 63(a)–(c) shows the initial and two subsequent simplex tableaus. The value of the game, from Figure 63(c), is $\frac{35}{12}$, which means that the value of the original game is $\frac{35}{12} - 3 = -\frac{1}{12}$. Optimal strategies agree with the answer stated in Exercise 3, Section 9.

EXAMPLE 3 Consider the following game: R and C simultaneously display 1, 2, or 3 pennies. If both show the same number of pennies, no money is exchanged; but if they show different numbers of pennies, R gets odd sums and C gets

	x_1	x_2	-1	
z_1	5	0	1	$= -y_1$
z_2	0	7	1	$= -y_2$
-1	1	1	0	$= f$
	$= u_1$	$= u_2$	$= g$	

(a)

	y_1	x_2	-1	
u_1	$\frac{1}{5}$	0	$\frac{1}{5}$	$= -x_1$
z_2	0	7	1	$= -y_2$
-1	$-\frac{1}{5}$	1	$\frac{1}{5}$	$= f$
	$= z_1$	$= u_2$	$= g$	

(b)

	y_1	y_2	-1	
u_1	$\frac{1}{5}$	0	$\frac{1}{5}$	$= -x_1$
u_2	0	$\frac{1}{7}$	$\frac{1}{7}$	$= -x_2$
-1	$-\frac{1}{5}$	$-\frac{1}{7}$	$\frac{12}{35}$	$= f$
	$= z_1$	$= z_2$	$= g$	

Figure 63 (c)

even sums. The matrix of the game is

		C shows 1	2	3
R shows	1	0	3	-4
	2	3	0	5
	3	-4	5	0

4	7	0
7	0	9
6	9	4

Since the second row has all nonnegative entries, the game is, if anything, in R's favor. And if R plays the first two rows with equal weight, his expectation is positive. Hence the value of the game is positive, and we do not have to add anything to the matrix entries. The simplex calculations needed to solve the game are shown in Figure 64(a)–(d). From this we see that the value of the game is $1/t = \frac{10}{7}$, and that optimal strategies are

$$p^0 = \frac{z^0}{t} = (\tfrac{5}{14}, \tfrac{4}{7}, \tfrac{1}{14}) \quad \text{and} \quad q^0 = \frac{x^0}{t} = \begin{pmatrix} \frac{5}{14} \\ \frac{4}{7} \\ \frac{1}{14} \end{pmatrix}.$$

	x_1	x_2	x_3	-1	
z_1	0	3	-4	1	$= -y_1$
z_2	3	0	5	1	$= -y_2$
z_3	-4	5	0	1	$= -y_3$
-1	1	1	1	0	$= f$
	$= u_1$	$= u_2$	$= u_3$	$= g$	

(a)

	y_2	x_2	x_3	-1	
z_1	0	3	-4	1	$= -y_1$
u_1	$\frac{1}{3}$	0	$\frac{5}{3}$	$\frac{1}{3}$	$= -x_1$
z_3	$\frac{4}{3}$	5	$\frac{20}{3}$	$\frac{7}{3}$	$= -y_3$
-1	$-\frac{1}{3}$	1	$\frac{2}{3}$	$\frac{1}{3}$	$= f$
	$= z_2$	$= u_2$	$= u_3$	$= g$	

(b)

	y_2	y_1	x_3	-1	
u_2	0	$\frac{1}{3}$	$-\frac{4}{3}$	$\frac{1}{3}$	$= -x_2$
u_1	$\frac{1}{3}$	0	$\frac{5}{3}$	$\frac{1}{3}$	$= -x_1$
z_3	$\frac{4}{3}$	$-\frac{5}{3}$	$\frac{40}{3}$	$\frac{1}{3}$	$= -y_3$
-1	$-\frac{1}{3}$	$-\frac{1}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$= f$
	$= z_2$	$= z_1$	$= u_3$	$= g$	

(c)

	y_2	y_1	y_3	-1	
u_2	$\frac{2}{15}$	$\frac{1}{6}$	$\frac{1}{10}$	$\frac{2}{5}$	$= -x_2$
u_1	$\frac{1}{6}$	$\frac{5}{24}$	$-\frac{1}{8}$	$\frac{1}{4}$	$= -x_1$
u_3	$\frac{1}{10}$	$-\frac{1}{8}$	$\frac{3}{40}$	$\frac{1}{20}$	$= -x_3$
-1	$-\frac{2}{5}$	$-\frac{1}{4}$	$-\frac{1}{20}$	$\frac{7}{10}$	$= f$
	$= z_2$	$= z_1$	$= z_3$	$= g$	

(d)

Figure 64

The reader should check that these strategies do solve the game.

The examples just solved could have been worked directly, without the use of the simplex method. However, the simplex method works just as well for much larger games for which there is no easy direct method.

EXERCISES

1. If $q^0 = x^0/t$, where x^0 solves the maximum problem stated in (1), and $t = ex^0$, show that q^0 is an optimal strategy for player C in the matrix game G.
2. Solve the following games by the simplex method.

(a)

1	0	3
−2	3	0
−4	5	−6

(b)

−2	3	0	5	−6
3	−4	5	0	7
−4	5	−6	7	0

3. Consider the matching-pennies game with matrix

$G =$

1	−1
−1	1

   (a) Substitute it directly into the simplex format described in rule (5) above, and show that the simplex method breaks down.
   (b) Consider the linear programming problem defined in (1) with this G and show directly that it has no solution.
4. Use the simplex method to derive formulas (3)–(7) of Section 9 for optimal strategies in a nonstrictly determined 2×2 game.
5. Rework Exercise 19 of Section 9 using the simplex method.
6. Rework Exercise 13 of Section 9 using the simplex method.
7. A passenger on a Mississippi riverboat was approached by a flashily dressed stranger (the gambler) who offered to play the following game: "You take a red ace and a black deuce and I'll take a red deuce and a black trey; we will simultaneously show one of the cards; if the colors don't match you pay me and if the colors match I'll pay you; moreover if you play the red ace we will exchange the difference of the numbers on the cards; but if you play the black deuce we will exchange the sum of the numbers. Since you will pay me with \$2 or \$4 if you lose and I will pay you either \$1 or \$5 if I lose, the game is obviously fair." Set up and solve the matrix game using the simplex method. Show that the game is not fair. Find the optimal strategies. [*Partial ans.* The gambler will win an average of 25 cents per game.]
8. Consider the following game: R chooses 0 or 1 and reveals his choice to C; C chooses 0 or 1, but does not reveal his choice to R; R then chooses 0 or 1 a second time. The sum of the three numbers chosen is computed and R receives odd sums while C receives even sums.
   (a) Show that R has four strategies: 00, 01, 10, 11.
   (b) Show that C has four strategies: (1) always choose 0, (2) choose the same as R, (3) choose opposite to R, (4) always choose 1.

(c) Show that the payoff matrix is

	(1)	(2)	(3)	(4)
00	0	0	1	1
01	1	1	−2	−2
10	1	−2	1	−2
11	−2	3	−2	3

(d) Solve the game by the simplex method, finding its value and optimal strategies.

$$[Ans.\ p^0 = (\tfrac{3}{4}, \tfrac{1}{4}, 0, 0),\ q^0 = \begin{pmatrix} \frac{3}{10} \\ \frac{9}{20} \\ \frac{1}{4} \\ 0 \end{pmatrix},\ v = \tfrac{1}{4}.]$$

9. Rework Exercise 8 assuming that the players choose 1 or 2 each time.

10. *The Silent Duel.* Two duelists each have a pistol that contains a single bullet and is equipped with a silencer. They advance toward each other in N steps, and each may fire at his opponent at the end of each step. Neither knows whether his opponent has fired, and each has but one shot in the game. The probability that a player will hit his opponent if he fires after moving k steps is k/N. A player gets 1 if he kills his opponent without being killed himself, -1 if he gets killed without killing his opponent, and 0 otherwise. Each player has N strategies corresponding to firing after steps $1, 2, \ldots, N$. Let i be the strategy chosen by R and let j be the strategy chosen by C.

(a) If $i < j$, show that the expected payoff to R is given by

$$\frac{N(i-j) + ij}{N^2}.$$

(b) If $i > j$, show that the expected payoff to R is given by

$$\frac{N(i-j) - ij}{N^2}.$$

(c) If $i = j$, show that the expected payoff to R is 0.

11. In Exercise 10, prove that the game is strictly determined and fair for $N = 2, 3$, and 4. Show that the optimal strategy for $N = 3, 4$ is to fire at the end of the second step in each case. For $N = 2$, show that any strategy is optimal.

12. In Exercise 10, show that the game is nonstrictly determined and fair for $N = 5$, and find the optimal strategies.
[*Ans.* $p^0 = (0, \frac{5}{11}, \frac{5}{11}, 0, \frac{1}{11})$, and q^0 is the column vector having the same components.]

13. A *symmetric matrix game* G is one for which $g_{ij} = -g_{ji}$ for i, $j = 1, 2, \ldots, n$. In other words, for every payment from C to R there is an equal payment from R to C. Show that every symmetric game is fair. [*Hint:* Show that if x^0 is optimal for R, then the column vector y with $y_k = x_k^0$ for $k = 1, \ldots, n$ is optimal for C. From this deduce that the value of the game is zero.]
14. Use Exercise 13 to show that the silent duel is fair for every N.
15. Consider a matrix game G with positive value in which the first row strictly dominates the second row. Show that in the simplex algorithm no entry in the second row will ever be chosen as a pivot in the first step.
16. Consider a matrix game G with positive value in which the first column dominates the second column. Show that if the pivot is chosen in the first column, after the end of the first simplex calculation the indicator for the second column will be nonnegative.

12 COMPUTER APPLICATIONS

The simplex method is ideal for solving linear programming problems, but it is difficult to carry it out by manual calculations. Actually, the method was designed for computers, and we shall show how to program it in BASIC.

The program LP follows Figure 18 closely. It starts with a DIM statement that allows a matrix A of 20 rows and 20 columns, and hence a 21×21 tableau. In the body of the program lines 100–1199 correspond to the 11 boxes in the figure, with each box starting at a new multiple of 100. Thus, for example, box 6 corresponds to the block of instructions starting at line 600. The remainder of the program prints answers and contains the DATA.

LP

```
10   DIM T(21,21),M(20,4),X(20),Y(20),U(20),V(20)
20
100  READ M,N
110  MAT READ T(M+1,N+1)
120  FOR J = 1 TO N
130  LET M(J,1) = M(J,3) = J
140  NEXT J
150  FOR I = 1 TO M
160  LET M(I,2) = M(I,4) = -I
170  NEXT I
180
200  FOR J = 1 TO N
210  IF T(M+1,J) > 1E-6 THEN 300
220  NEXT J
230  GOTO 1100
240
300  REM  J IS THE PIVOTAL COLUMN
310
400  FOR I = 1 TO M
```

```
410 IF T(I,J) > 1E-6 THEN 600
420 NEXT I
430
500 PRINT "MAXIMUM PROBLEM HAS UNBOUNDED SOLUTION."
510 PRINT "MINIMUM PROBLEM HAS NO SOLUTION."
520 GOTO 1999
530
600 LET M1 = 1E30
610 FOR I = 1 TO M
620 IF T(I,J) < 1E-6 THEN 680
630 LET Q = T(I,N+1)/T(I,J)
640 IF Q >= M1 THEN 680
650 LET M1 = Q
660 LET I1 = I
680 NEXT I
690 LET I = I1
699
700 LET C = T(I,J)
710 FOR J1 = 1 TO N+1
715 IF J1 = J THEN 730
720 LET T(I,J1) = T(I,J1)/C
730 NEXT J1
740
800 FOR I1 = 1 TO M+1
810 IF I1 = I THEN 860
820 LET C = T(I1,J)
830 FOR J1 = 1 TO N+1
835 IF J1 = J THEN 850
840 LET T(I1,J1) = T(I1,J1) - C*T(I,J1)
850 NEXT J1
860 NEXT I1
870
900 LET C = T(I,J)
910 LET T(I,J) = 1/C
920 FOR I1 = 1 TO M+1
930 IF I1 = I THEN 970
940 LET T(I1,J) = -T(I1,J)/C
970 NEXT I1
980

1000 LET A = M(J,1)
1010 LET M(J,1) = M(I,2)
1020 LET M(I,2) = A
1030 LET A = M(J,3)
1040 LET M(J,3) = M(I,4)
1050 LET M(I,4) = A
1060 GOTO 200
1070
1100 MAT X = ZER(N)
1105 MAT Y = ZER(M)
1110 MAT U = ZER(N)
1115 MAT V = ZER(M)
```

```
1120 FOR I = 1 TO M
1125 LET A = T(I,N+1)
1130 LET S = M(I,2)
1135 IF S<0 THEN 1150
1140 LET X(S) = A
1145 GOTO 1155
1150 LET Y(-S) = A
1155 NEXT I
1160 FOR J = 1 TO N
1165 LET A = -T(M+1,J)
1170 LET S = M(J,3)
1175 IF S<0 THEN 1190
1180 LET U(S) = A
1185 GOTO 1195
1190 LET V(-S) = A
1195 NEXT J
1199
1210 PRINT "VALUE = "; -T(M+1,N+1)
1220 PRINT "X = ";
1230 MAT PRINT X;
1240 PRINT "Y = ";
1250 MAT PRINT Y;
1260 PRINT "U = ";
1270 MAT PRINT U;
1280 PRINT "V = ";
1290 MAT PRINT V;
1299
1900 DATA 2,3
1910 DATA 6,2,4,200
1911 DATA 2,2,12,160
1912 DATA 12,8,24,0
1920
1999 END

READY

RUN

LP

VALUE =  680.
X =
 10.  70.  0
Y =
 0  0
U =
 0  0  16.
V =
 1.  3.

0.279 SEC.
READY
```

Since the flow diagram is, in effect, an explanation of the program, only a few comments are necessary. We have used only one tableau T, and hence the changes are made within the tableau, rather than copying it over. This is possible as long as we are careful in boxes 7 and 8 not to change the entries of the pivotal column. Then the original entries are still available for box 9. Also, in testing for positive entries we have elected to check whether the entry is greater than 10^{-6}, to avoid round-off errors.

The only major change from Figure 18 to the program LP is the matrix M. We have to keep track of where the various variables are placed around the margin of the tableau. The matrix M has four columns, corresponding to the four margins. The first column keeps track of the variables on top, the second column of the right side, the third of the bottom, and the fourth of the left-side margin. In each case the entries are the subscripts of the variables, except that negative entries indicate that we have y or v rather than x or u. For example, if M(3,2) = 1, then in the third row the right-hand marginal is x_1; while if M(3,2) = -1, then it is y_1. Similarly, if M(1,3) = 4, then the first variable on the bottom is u_4, while a -4 would indicate v_4.

The DATA in LP is taken from Example 2 in Section 5, and the RUN shows the results previously obtained. By changing the DATA we work out two other previous examples: LP2 corresponds to Example 3, and LP3 to Example 4—which has no solution.

```
1900 DATA 2,2
1910 DATA 2,1,3
1911 DATA 3,1,4
1912 DATA 17,5,0

READY

RUN

LP2

VALUE =   22.6667
X =
 1.33333   0
Y =
 0.333333   0
U =
 0   0.666667
V =
 0   5.66667

0.242 SEC.
READY
```

```
1900 DATA 2,2
1910 DATA -1,1,1
1911 DATA 1,-1,1
1912 DATA 1,1,Ø

READY

RUN

LP3

MAXIMUM PROBLEM HAS UNBOUNDED SOLUTION.
MINIMUM PROBLEM HAS NO SOLUTION.

Ø.226 SEC.
READY
```

To show the power of the computerized simplex method we need a larger example. The program LP4 solves the following maximum problem. The

```
1900 DATA 6,3
1910 DATA .8,.5,.3,82
1911 DATA .15,.2,.2,3Ø
1912 DATA .Ø5,.2,.2,3Ø
1913 DATA Ø,.Ø5,.15,16
1914 DATA Ø,.Ø5,.1,1Ø
1915 DATA Ø,Ø,.Ø5,4
1916 DATA 25,4Ø,75,Ø

READY

RUN

LP4

VALUE =  86ØØ.
X =
 4Ø.  4Ø.  8Ø
Y =
 6.  Ø  4.  2.  Ø  Ø
U =
 Ø  Ø  Ø
V =
 Ø  166.667  Ø  Ø  133.333  566.667

Ø.314 SEC.
READY
```

Tasty Nut Company packages three kinds of boxes of mixed nuts. Each box contains a pound of nuts, according to the following rules:

Cheap mix:	80% peanuts, 15% almonds, 5% cashews.
Medium mix:	50% peanuts, 20% almonds, 20% cashews, 5% walnuts, 5% hazelnuts.
Fancy mix:	30% peanuts, 20% almonds, 20% cashews, 15% walnuts, 10% hazelnuts, 5% Brazil nuts.

The manufacturer has on hand 82 lb of peanuts, 30 lb almonds, 30 lb cashews, 16 lb walnuts, 10 lb hazelnuts, and 4 lb Brazil nuts. If he makes a profit of 25 cents on the simple mix, 40 cents on the medium mix, and 75 cents on the fancy mix, how much of each should he package? The program LP4 contains the appropriate tableau and solves the problem. We find a value of 8600, i.e., a profit of \$86. From the vector X we note that the optimal solution is to package 40 lb each of the two cheaper mixes, and 80 lb of the fancy mix. From Y we note that we shall have left over 6 lb of peanuts and smaller amounts of cashews and walnuts. The vector V is particularly interesting. It imputes to almonds, hazelnuts, and Brazil nuts per pound values of \$$1\frac{2}{3}$, \$$1\frac{1}{3}$, and \$$5\frac{2}{3}$, respectively. This makes it very tempting to buy more Brazil nuts.

The program LP5 differs from LP4 only in adding $\frac{1}{2}$ lb of Brazil nuts to the stock. We note that the profit has increased by one-half of the imputed

```
1915 DATA Ø,Ø,.Ø5,4.5
RUN

LP5

VALUE =   8883.33
X =
 53.3333  2Ø.   9Ø
Y =
 2.33333  Ø   5.33333   1.5   Ø   Ø
U =
 Ø  Ø  Ø
V =
 Ø   166.667   Ø   Ø   133.333   566.667

Ø.356 SEC.
READY
```

value of \$$5\frac{2}{3}$, which is what we would expect from adding $\frac{1}{2}$ lb. It is interesting to note that so small a change in the inventory results in the drastic change in the optimal production schedule.

The program LP6 illustrates the fact that the interpretation of imputed

```
1915 DATA Ø,Ø,.Ø5,5
RUN
```

```
LP6

VALUE =   9125
X =
 65.   Ø   1ØØ.
Y =
 Ø  Ø.25   6.75   1   Ø   2.89557 E-8
U =
 Ø   8.4375   Ø
V =
 31.25   Ø   Ø   Ø   656.25   Ø

Ø.331 SEC.
READY
```

values is correct only as long as the nature of the solution does not change. It differs from the original LP4 by adding one full pound of Brazil nuts. However, the profit goes up only $\$5\frac{1}{4}$, less than the imputed value. We find an explanation for this by noting that now peanuts have become a critical commodity instead of Brazil nuts. And we no longer produce the medium mix. The nature of the dual solution changes somewhere between 4.5 and 5 lb of Brazil nuts, and hence the imputed value holds for only part of the change.

We next consider some computer applications to the solution of matrix games. The program STRICT tests whether a given matrix game is strictly determined, and if it is, it finds the saddle values. We first read the game matrix. Then we set V to an incorrect value. If during the computation we discover a saddle value, we shall then know the value of the game and reset V. If V is not reset during the program, then we shall know that the game is not strictly determined.

The strategy of the program is to look at every entry in the matrix to see whether it is a saddle value. We start the double loop, which will run through all rows and all columns, at line 100. At line 150 we ignore the entry if it is different from the value of the game. However, if we have not yet found the value of the game this test is inapplicable, and that is the reason for line 140. In the loop at lines 210–230 we reject the entry if there is a smaller entry on the same row. In the loop at lines 260–280 we reject it if there is a larger entry in the same column. If it passes both of these tests, it is a saddle value. At lines 310–330 we set V to the correct value, print the saddle value, and go on to the next entry. At lines 610–630 we either print the value of the game or we indicate that the game is not strictly determined.

The data for the first RUN is from Example 3 in Section 8. This is a

```
STRICT

10 READ M,N
20 MAT READ G(M,N)
30 LET V = -99999
40
100 REM  LOOK AT EVERY ENTRY
110 FOR I = 1 TO M
120 FOR J = 1 TO N
130 LET V1 = G(I,J)
140 IF V = -99999 THEN 200
150 IF V1 <> V THEN 400
160
200 REM  IS IT A ROW MIN
210 FOR J1 = 1 TO N
220 IF G(I,J1) < V1 THEN 400
230 NEXT J1
240
250 REM  IS IT A COLUMN MAX
260 FOR I1 = 1 TO M
270 IF G(I1,J) > V1 THEN 400
280 NEXT I1
290
300 REM  IT IS A SADDLE
310 LET V = V1
320 PRINT "ROW";I;"- COLUMN";J;"IS A SADDLE"
330 GOTO 500
340
400 REM  NOT A SADDLE, IGNORE
410
500 REM  NEXT ENTRY
510 NEXT J
520 NEXT I
530
600 REM  IS IT STRICTLY DETERMINED
610 IF V = -99999 THEN 650
620 PRINT "STRICTLY DETERMINED. VALUE =";V
630 GOTO 999
650 PRINT "NOT STRICTLY DETERMINED."
660
900 DATA 3,3
910 DATA 50,50,80
911 DATA 50,50,80
912 DATA 20,20,50
999 END
READY
```

```
RUN

STRICT

ROW 1 - COLUMN 1 IS A SADDLE
ROW 1 - COLUMN 2 IS A SADDLE
ROW 2 - COLUMN 1 IS A SADDLE
ROW 2 - COLUMN 2 IS A SADDLE
STRICTLY DETERMINED. VALUE = 50

0.125 SEC.
READY

900 DATA 2,2
910 DATA 0,1
911 DATA 2,0
RUN

STRICT

NOT STRICTLY DETERMINED.

0.110 SEC.
READY
```

strictly determined game. For the second RUN we have chosen the example of Figure 55(a), which is not a strictly determined game.

The most interesting computer application to games is modifying the program LP to apply to matrix games. If in our DATA we have the enlarged game matrix, including a last column and last row of all 1's (except for a zero in the lower right-hand corner), then we know from Section 11 that we may apply the simplex method directly. The only change necessary to our program LP is to modify the output, since the value is $1/t$, and the vectors X and V have to be multiplied by this value. In the program GAME

```
GAME

1200 LET T = -T(M+1,N+1)
1210 PRINT "VALUE = ";1/T
1220 PRINT "STRATEGY FOR R:";
1230 MAT V = (1/T)*V
1240 MAT PRINT V;
1250 PRINT "STRATEGY FOR C:";
1260 MAT X = (1/T)*X
1270 MAT PRINT X;
```

```
1299
1900 DATA 3,3
1910 DATA 0,3,-4,1
1911 DATA 3,0,5,1
1912 DATA -4,5,0,1
1913 DATA 1,1,1,0
1920
1999 END
READY

RUN

GAME

VALUE =  1.42857
STRATEGY FOR R:
 0.357143   0.571429   7.14286 E-2
STRATEGY FOR C:
 0.357143   0.571429   7.14286 E-2

0.269 SEC.
READY
```

we show only the modifications we make, starting with line 1200. As our example we have used Example 3 of Section 11. We obtain the same results as in the text.

A computer version of the simplex method allows one to solve both very large linear programming problems and very large matrix games.

EXERCISES

Only Exercises 5–12 require the use of a computer.

1. Modify the program LP so that it will print the current tableau after each iteration. Be sure to print the variables in the margins.
2. In the RUN of LP5, for the optimal solution X(1) = 53.3333, what should the manufacturer do if he produces only one-pound boxes?
3. The program GAME works only if the value is positive. Modify it so that if there are any negative entries it will add a sufficiently large positive number to all entries. [*Hint:* If you do this, the value of the game changes.]
4. Describe an alternate strategy for the program STRICT. Which do you think is faster?
5. Use LP to solve the tableaus in Figures 34 and 35 (Section 6).
6. Use LP to solve Exercises 12, 13, and 14 in Section 5.
7. How does the optimal strategy of the Tasty Nut Company change if

the profits on the three kinds of mixes are 30, 50, and 90 cents, respectively? [*Ans.* It does not change.]

8. Redo Exercise 7 for profits of
   (a) 20, 30, and 40 cents.
   (b) 25, 40, and 40 cents.
9. Use STRICT on the games in Exercise 2 of Section 8.
10. Use STRICT on the games in Exercise 3 of Section 8.
11. Try an alternate program for STRICT (see Exercise 4) to see whether it is faster.
12. Try the program of Exercise 3 on the following game:

3	−1	0
−11	0	5
0	−7	7
8	3	−9

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