Sets and Counting Problems

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1 INTRODUCTION

A well-defined collection of objects is known as a *set*. This concept, in its complete generality, is of great importance in mathematics since all of mathematics can be developed by starting from it.

The various pieces of furniture in a given room form a set. So do the books in a given library, or the integers between 1 and 1,000,000 or all the ideas that mankind has had, or the human beings alive between 1 billion B.C. and A.D. 10 billion. These examples are all examples of *finite* sets, that is, sets having a finite number of elements. All the sets discussed in this book will be finite sets.

The collection of all tall people is *not* a well-defined set, because the word "tall" is not precisely defined. On the other hand the set of all people whose height is six feet or more *is* a well-defined set, because we can determine whether any given person belongs to the set simply by measuring his height.

There are two essentially different ways of specifying a set. One can give a rule by which it can be determined whether or not a given object is a member of the set, or one can give a complete list of the elements in the set. We shall say that the former is a *description* of the set and the latter is a *listing* of the set. For example, we can define a set of four people as (a) the members of the string quartet which played in town last night, or (b) four particular persons whose names are Jones, Smith, Brown, and Green. It is customary to use braces to surround the listing of a set; thus the set above should be listed {Jones, Smith, Brown, Green}.

We shall frequently be interested in sets of logical possibilities, since the analysis of such sets is very often a major task in the solving of a problem. Suppose, for example, that we were interested in the successes of three candidates who enter the presidential primaries (we assume there are no other entries). Suppose that the key primaries will be held in New Hampshire, Minnesota, Winsonsin, and California. Assume that candidate A enters all the primaries, that B does not contest in New Hampshire's primary, and C does not contest in Wisconsin's. A list of the logical possibilities is given in Figure 1. Since the New Hampshire and Wisconsin primaries can

Possibility Number	Winner in New Hampshire	Winner in Minnesota	Winner in Wisconsin	Winner in California
P1	A	A	A	A
P2	A	A	A	B
P3	A	A	A	C
P4	A	A	B	A
P5	A	A	B	B
P6	A	A	B	C
P7	A	B	A	A
P8	A	B	A	B
P9	A	B	A	C
P10	A	B	B	A
P11	A	B	B	B
P12	A	B	B	C
P13	A	C	A	A
P14	A	C	A	B
P15	A	C	A	C
P16	A	C	B	A
P17	A	C	B	B
P18	A	C	B	C
P19	C	A	A	A
P20	C	A	A	B
P21	C	A	A	C
P22	C	A	B	A
P23	C	A	B	B
P24	C	A	B	C
P25	C	B	A	A
P26	C	B	A	B
P27	C	B	A	C
P28	C	B	B	A
P29	C	B	B	B
P30	C	B	B	C
P31	C	C	A	A
P32	C	C	A	B
P33	C	C	A	C
P34	C	C	B	A
P35	C	C	B	B
P36	C	C	B	C

Figure 1

each end in two ways, and the Minnesota and California primaries can each end in three ways, there are in all $2 \cdot 2 \cdot 3 \cdot 3 = 36$ different logical possibilities as listed in Figure 1.

A set that consists of some members of another set is called a *subset* of that set. For example, the set of those logical possibilities in Figure 1 for which the statement "Candidate A wins at least three primaries" is true, is a subset of the set of all logical possibilities. This subset can also be defined by listing its members: {P1, P2, P3, P4, P7, P13, P19}.

In order to discuss all the subsets of a given set, let us introduce the following terminology. We shall call the original set the *universal set,* one-element subsets will be called *unit sets,* and the set which contains no members the *empty set*. We do not introduce special names for other kinds of subsets of the universal set. As an example, let the universal set $\mathcal{U}$ consist of the three elements $\{a, b, c\}$. The *proper subsets* of $\mathcal{U}$ are those sets containing some but not all of the elements of $\mathcal{U}$. The proper subsets here consist of three two-element sets—namely, $\{a, b\}$, $\{a, c\}$, and $\{b, c\}$—and three unit sets—namely, $\{a\}$, $\{b\}$, and $\{c\}$. To complete the picture, we also consider the universal set a subset (but not a proper subset) of itself, and we consider the empty set* $\mathcal{E}$, which contains no elements of $\mathcal{U}$, as a subset of $\mathcal{U}$. At first it may seem strange that we should include the sets $\mathcal{U}$ and $\mathcal{E}$ as subsets of $\mathcal{U}$, but the reasons for their inclusion will become clear later.

We saw that the three-element set above had $8 = 2^3$ subsets. In general, a set with n elements has 2^n subsets, as can be seen in the following manner. We form subsets P of $\mathcal{U}$ by considering each of the elements of $\mathcal{U}$ in turn and deciding whether or not to include it in the subset P. If we decide to put every element of $\mathcal{U}$ into P, we get the universal set, and if we decide to put no element of $\mathcal{U}$ into P, we get the empty set. In most cases we shall put some but not all the elements into P and thus obtain a proper subset of $\mathcal{U}$. We have to make n decisions, one for each element of the set, and for each decision we have to choose between two alternatives. We can make these decisions in $2 \cdot 2 \cdot \ldots \cdot 2 = 2^n$ ways, and hence this is the number of different subsets of $\mathcal{U}$ that can be formed. Observe that our formula would not have been so simple if we had not included the universal set and the empty set as subsets of $\mathcal{U}$.

In the example of the voting primaries above there are 2^{36} or about 70 billion subsets. Of course, we cannot deal with this many subsets in a practical problem, but fortunately we are usually interested in only a few of the subsets. The most interesting subsets are those which can be defined by means of a simple rule such as "the set of all logical possibilities in which C loses at least two primaries." It would be difficult to give a simple description for the subset containing the elements {P1, P4, P14, P30, P34}. On the other hand, we shall see in the next section how to define new subsets in terms of subsets already defined.

*Many books use ϕ to symbolize the empty set.

EXAMPLES We illustrate the two different ways of specifying sets in terms of the primary voting example. Let the universal set $\mathcal{U}$ be the logical possibilities given in Figure 1.

1. What is the subset of $\mathcal{U}$ in which candidate B wins more primaries than either of the other candidates? *Answer:* {P11, P12, P17, P23, P26, P28, P29}.

2. What is the subset in which the primaries are split two and two? *Answer:* {P5, P8, P10, P15, P21, P30, P31, P35}.

3. Describe the set {P1, P4, P19, P22}. *Answer:* The set of possibilities for which A wins in Minnesota and California.

4. How can we describe the set {P18, P24, P27}? *Answer:* The set of possibilities for which C wins in California, and the other primaries are split three ways.

EXERCISES

1. In the primary example, list each of the following sets.
- **(a)** The set in which A and C win the same number of primaries.
- **(b)** The set in which the winner of the New Hampshire primary does not win another primary.
- **(c)** The set in which C wins all four primaries.

2. Again referring to the primary example, give simple descriptions of the following sets.
- **(a)** [P1, P4, P8, P11, P15, P18, P19, P22, P26, P29, P33, P36].
- **(b)** [P18, P22, P26].
- **(c)** [P1, P11, P19, P29].

3. The primaries are considered decisive if a candidate can win three primaries, or if he wins two primaries including California. List the set in which the primaries are decisive.

4. List the set of four-letter "words" formed by writing down the letters of the word *stop* in all possible ways. [*Hint:* The set has 24 elements.]

5. In Exercise 4, list the following subsets:
- **(a)** The set of English words. [*Partial Ans*. There are 6.]
- **(b)** The set in which the letters are in alphabetical order either from left to right or from right to left.
- **(c)** The set in which *p* and *t* are next to each other.
- **(d)** The set in which only *s* is between *o* and *t*.
- **(e)** The set in which *t* and *s* are at the ends.

6. Find all pairs in Exercise 5 in which one set is a subset of the other.

7. A baker has four feet of display space to fill with some combination of bread, cake, and pie. A loaf of bread takes one-half foot of space, a cake takes one foot, and a pie takes two feet. Construct the set of possible distributions of shelf space, considering only the *total* space allotted to each kind of item.

8. In Exercise 7, list the following subsets.

(a) The set in which as much space is devoted to pie as to cake.
(b) The set in which equal space is given to two different items, and at least two different items are displayed.
(c) The set in which six or more items are displayed.
(d) The set in which at least two of the above conditions are satisfied.

9. A man has 65 cents in change, but he has no pennies and has at least as many dimes as nickels. Find the set of possibilities for his collection of coins.
10. In Exercise 9, list the following subsets.
    (a) The set in which the man has exactly one quarter.
    (b) The set in which the man has more half-dollars than quarters.
    (c) The set in which the man has fewer than six coins.
    (d) The set in which none of the above conditions is satisfied.
11. A set has 51 elements. How many subsets does it have? How many of the subsets have an even number of elements? [*Ans.* 2^{51}, 2^{50}.]
12. Do Exercise 11 for the case of a set with 52 elements.

2 OPERATIONS ON SUBSETS

In Chapter 1 we considered the ways in which one could form new statements from given statements. Now we shall consider an analogous procedure, the formation of new sets from given sets. We shall assume that each of the sets that we use in the combination is a subset of some universal set, and we shall also want the newly formed set to be a subset of the same universal set. As usual, we can specify a newly formed set either by a description or by a listing.

If P and Q are two sets, we shall define a new set $P \cap Q$, called the *intersection* of P and Q as follows: $P \cap Q$ is the set which contains those and only those elements which belong to both P and Q. As an example, consider the logical possibilities listed in Figure 1. Let P be the subset in which candidate A wins at least three primaries, i.e., the set {P1, P2, P3, P4, P7, P13, P19}; let Q be the subset in which A wins the first two primaries, i.e., the set {P1, P2, P3, P4, P5, P6}. Then the intersection $P \cap Q$ is the set in which both events take place, i.e., where A wins the first two primaries *and* wins at least three primaries. Thus $P \cap Q$ is the set {P1, P2, P3, P4}.

If P and Q are two sets, we shall define a new set $P \cup Q$ called the *union* of P and Q as follows: $P \cup Q$ is the set that contains those and only those elements that belong either to P or to Q (or to both). In the example in the paragraph above, the union $P \cup Q$ is the set of possibilities for which either A wins the first two primaries *or* wins at least three primaries, i.e., the set {P1, P2, P3, P4, P5, P6, P7, P13, P19}.

To help in visualizing these operations we shall draw diagrams, called *Venn diagrams,** which illustrate them. We let the universal set be a rectangle and let subsets be circles drawn inside the rectangle. In Figure 2 we show

*Named after the English logician John Venn (1834–1923).

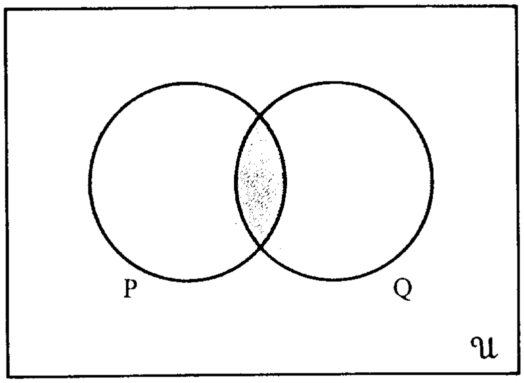


Figure 2

two sets P and Q as shaded circles, P shaded in color and Q in gray. Then the area shaded in both color and gray is the intersection $P \cap Q$ and the total shaded area is the union $P \cup Q$.

If P is a given subset of the universal set $\mathcal{U}$, we can define a new set $\tilde{P}$ called the *complement* of P as follows: $\tilde{P}$ is the set of all elements of $\mathcal{U}$ that are *not* contained in P. For example, if, as above, Q is the set in which candidate A wins the first two primaries, then $\tilde{Q}$ is the set {P7, P8, . . ., P36}. The shaded area in Figure 3 is the complement of the set P. Observe that the complement of the empty set $\mathcal{E}$ is the universal set $\mathcal{U}$, and also that the complement of the universal set is the empty set.

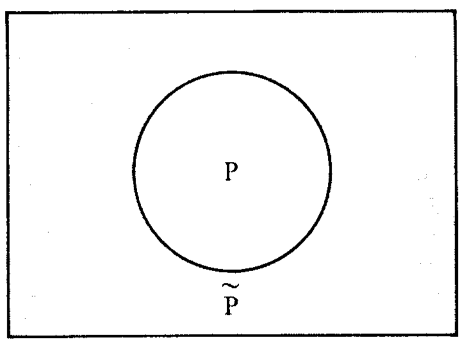


Figure 3

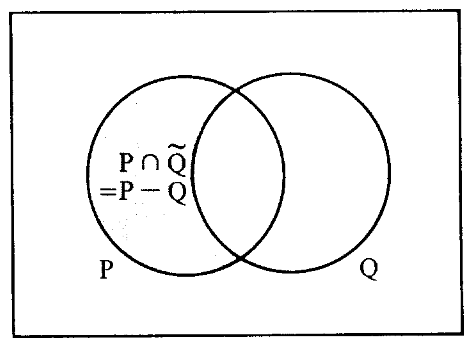


Figure 4

Sometimes we shall be interested in only part of the complement of a set. For example, we might wish to consider the part of the complement of the set Q that is contained in P, i.e., the set $P \cap \tilde{Q}$. The shaded area in Figure 4 is $P \cap \tilde{Q}$.

A somewhat more suggestive definition of this set can be given as follows: Let $P - Q$ be the *difference* of P and Q, that is, the set that contains those elements of P that do not belong to Q. Figure 4 shows that $P \cap \tilde{Q}$ and $P - Q$ are the same set. In the primary voting example above, the set $P - Q$ can be listed as {P7, P13, P19}.

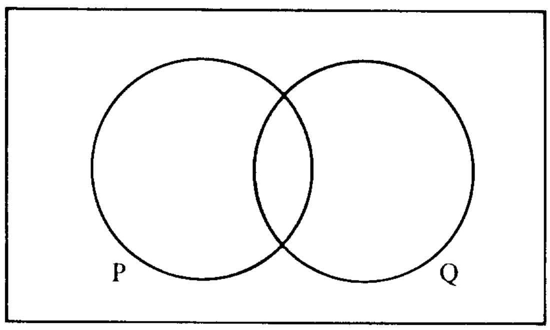
The complement of a subset is a special case of a difference set, since we can write $\tilde{Q} = \mathcal{U} - Q$. If P and Q are nonempty subsets whose intersection is the empty set, i.e., $P \cap Q = \mathcal{E}$, then we say that they are *disjoint* subsets.

EXAMPLE 1 In the primary voting example let R be the set in which A wins the first three primaries, i.e., the set {P1, P2, P3}; let S be the set in which A wins the last two primaries, i.e., the set {P1, P7, P13, P19, P25, P31}. Then $R \cap S = \{P1\}$ is the set in which A wins the first three primaries and also the last two, that is, he wins all the primaries. We also have

$$R \cup S = \{P1, P2, P3, P7, P13, P19, P25, P31\},$$

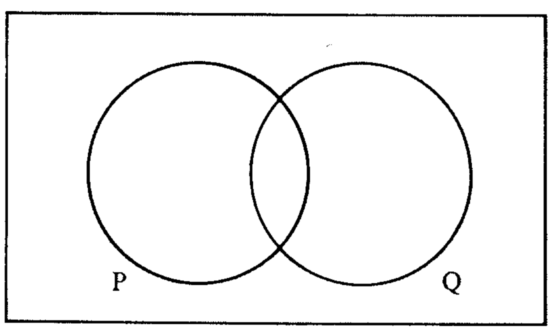
which can be described as the set in which A wins the first three primaries or the last two. The set in which A does not win the first three primaries is $\tilde{R} = \{P4, P5, \ldots, P36\}$. Finally, we see that the difference set $R - S$ is the set in which A wins the first three primaries but not both of the last two. This set can be found by taking from R the element P1 which it has in common with S, so that $R - S = \{P2, P3\}$.

EXAMPLE 2 Let us give a step-by-step construction of the Venn diagram for the set $(P \cap Q) \cup (\tilde{P} \cap \tilde{Q})$. Figure 5 shows the set $P \cap Q$ which is the same as



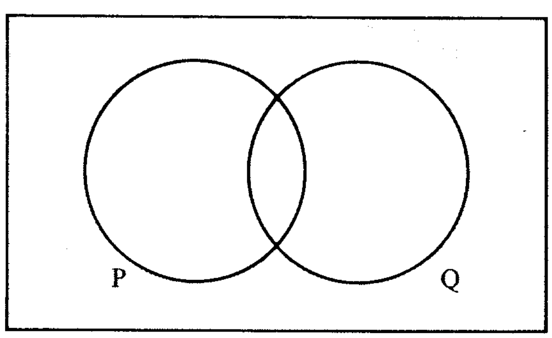
$P \cap Q$

Figure 5



$\tilde{P} \cap \tilde{Q}$

Figure 6



$(P \cap Q) \cup (\tilde{P} \cap \tilde{Q})$

Figure 7

the set of Figure 2 shaded in both color and gray; Figure 6 shows the set $\tilde{P} \cap \tilde{Q}$ which is the same as the complement of the shaded area in Figure 2. Finally, Figure 7 is the union of the two areas in Figures 5 and 6 and is the answer desired.

EXERCISES

1. Draw Venn diagrams for the following sets:
   (a) $P \cap Q$.
   (b) $\tilde{P} \cup Q$.
   (c) $P \cup \tilde{Q}$.
   (d) $\tilde{P} \cup \tilde{Q}$.
2. Give a step-by-step construction of the diagram for $((P \cup Q) - (P \cap Q)) \cap \tilde{Q}$.
3. Venn diagrams are also useful when three subsets are given. Construct such a diagram, given the subsets P, Q, and R. Identify each of the eight resulting areas in terms of P, Q, and R.
4. In assigning dormitory roomates, a college considers a student's sex, whether or not the student wants to live in a coed dorm, and whether the student is a freshman or an upperclassman. Draw a Venn diagram, and identify each of the eight areas.
5. Let F be the set of females, U the set of upperclassmen, and C the set of students desiring to live in a coed dorm. Define (symbolically) the following sets:
   (a) Upperclass males who do not want to live in a coed dorm.
   [*Ans.* $U \cap \tilde{F} \cap \tilde{C}$.]
   (b) Women who want to live in a coed dorm.
   (c) Male students who want to live in a coed dorm and are freshmen.
   (d) Women who are not freshmen and do not want to live in a coed dorm.
6. The college decides that two students can be roommates if both are of the same sex or if both are upperclassmen who want to live in a coed dorm. Identify the sets of students with the property that any two members of the set can be roommates.
7. The results of a survey of church attendance and golf playing are given in the following table:

Occupation	Golfs and Attends	Golfs and Doesn't Attend	Doesn't Golf and Attends	Doesn't Golf and Doesn't Attend
Doctor	15	20	3	2
Lawyer	10	9	9	6
CPA	8	0	11	7

Let D = doctor, L = lawyer, C = CPA, G = golfs, A = attends. Determine the number of people in each of the following classes.

(a) $D \cap G \cap \tilde{A}$.

(b) $\tilde{C} \cap \tilde{G} \cap A$.

(c) $\widetilde{(G \cup A)} \cap L$.

(d) $(D \cup L) \cap G$. [*Ans.* 54.]

(e) $\tilde{L} \cap ((A \cap G) \cup (\tilde{A} \cap G))$. [*Ans.* 43.]

8. In Exercise 7, which set of each of the following pairs has more members?

(a) $(D \cap G) - A$ or $\tilde{L} \cup (G \cap A)$?

(b) $\mathcal{E}$ or $C \cap \tilde{A} \cap G$?

(c) $\widetilde{(D \cup L)}$ or C?

9. A college student hired to survey 1000 beer drinkers and record their age, sex, and educational level turned in the following figures: 700 males, 600 people over 25 years of age, 400 college graduates, 250 male college graduates, 225 college graduates over 25, 350 males over 25, and 150 male college graduates over 25. After turning in his results, he was fired. Why? [*Hint:* Draw a Venn diagram with three circles—for males, college graduates, and those over 25. Fill in the numbers in each of the eight areas, using the data given above. Start from the end of the list and work back.]

10. A survey of 110 lung cancer patients showed that 70 were cigarette smokers, 60 lived in urban areas, and 35 had hazardous occupations. Forty of the smokers lived in urban areas, 15 had hazardous occupations, and 5 were in both categories. Ten of the patients with hazardous occupations neither lived in an urban area nor smoked.

(a) How many of the patients living in urban areas had hazardous occupations? [*Ans.* 15.]

(b) How many of those living in the urban areas neither smoked nor had hazardous occupations? [*Ans.* 10.]

(c) How many patients smoke if and only if they live in an urban area?

(d) How many patients neither smoked, nor lived in an urban area, nor had a hazardous occupation?

11. A second survey of 100 patients had the following results: 45 smokers who lived in urban areas, 37 of whom did not have a hazardous occupation; 20 people with hazardous occupations, of whom 10 live in urban areas and 10 smoke; 75 smokers; and 10 who neither smoke, nor have a hazardous occupation, nor live in an urban area.

(a) How many patients with hazardous occupations neither smoke nor live in an urban area? [*Ans.* 8]

(b) How many patients live in an urban area?

(c) How many patients smoke if and only if they do not have a hazardous occupation?

(d) How many patients smoke, have a hazardous occupation, and live in an urban area?

12. The following table summarizes the responses of 100 students asked what they thought about during math lectures:

Class and Status	Neither Food Nor Football	Only Food	Only Football	Food and Football
Senior Majors	20	12	4	6
Senior Nonmajors	8	10	15	0
Junior Majors	2	1	6	1
Junior Nonmajors	3	5	5	2

All the categories can be defined in terms of the following four: M (majors), S (seniors), F (food), and FT (football). How many students fall into each of the following categories?

(a) S
(b) $S - M$
(c) $M - S$
(d) $J \cap \tilde{M} \cap F \cap FT$
(e) $(J \cap F)$
(f) $\tilde{J} \cup \tilde{F}$ [*Ans.* 91.]
(g) $S \cap \tilde{M} \cap F$
(h) $(S \cup F) - \widetilde{FT}$ [*Ans.* 28.]
(i) $S \cap M \cap (\widetilde{F \cup FT})$ [*Ans.* 20.]
(j) $S \cup J$

3 THE RELATIONSHIP BETWEEN SETS AND COMPOUND STATEMENTS

The reader may have observed several times in the preceding sections that there was a close connection between sets and statements, and between set operations and compounding operations. In this section we shall formalize these relationships.

If we have a number of statements relative to a set of logical possibilities, there is a natural way of assigning a set to each statement. First we take the set of logical possibilities as our universal set. Then to each statement we assign the subset of logical possibilities of the universal set for which that statement is true. This idea is so important that we embody it in a formal definition.

Definition Let $\mathcal{U}$ be a set of logical possibilities, let p be a statement relative to it, and let P be that subset of the possibilities for which p is true; then we call P the *truth set* of p.

If p and q are statements, then $p \vee q$ and $p \wedge q$ are also statements and hence must have truth sets. To find the truth set of $p \vee q$, we observe that it is true whenever p is true or q is true (or both). Therefore we must assign to $p \vee q$ the logical possibilities which are in P or in Q (or both); that is, we must assign to $p \vee q$ the set $P \cup Q$. On the other hand, the statement

$p \wedge q$ is true only when both p and q are true, so that we must assign to $p \wedge q$ the set $P \cap Q$.

Thus we see that there is a close connection between the logical operation of disjunction and the set operation of union, and also between conjunction and intersection. A careful examination of the definitions of union and intersection shows that the word "or" occurs in the definition of union and the word "and" occurs in the definition of intersection. Thus the connection between the two theories is not surprising.

Since the connective "not" occurs in the definition of the complement of a set, it is not surprising that the truth set of $\sim p$ is $\tilde{P}$. This follows since $\sim p$ is true when p is false, so that the truth set of $\sim p$ contains all logical possibilities for which p is false, that is, the truth set of $\sim p$ is $\tilde{P}$.

The truth sets of two propositions p and q are shown in Figure 8. Also marked on the diagram are the various logical possibilities for these two statements. The reader should pick out in this diagram the truth sets of the statements $p \vee q$, $p \wedge q$, $\sim p$, and $\sim q$.

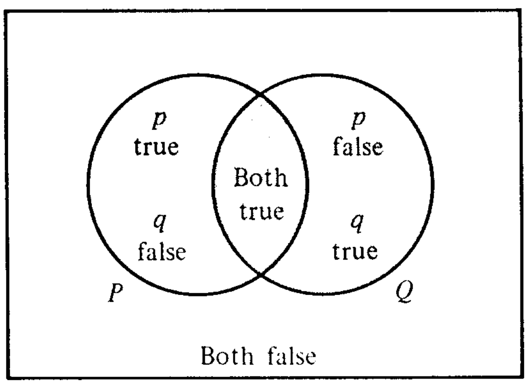


Figure 8

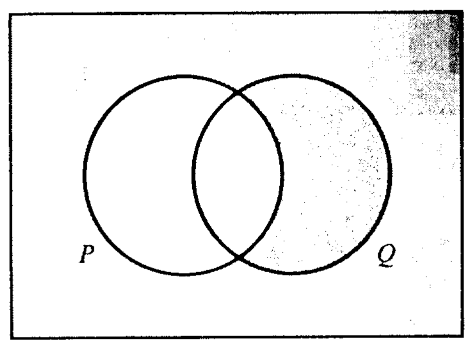


Figure 9

The connection between a statement and its truth set makes it possible to "translate" a problem about compound statements into a problem about sets. It is also possible to go in the reverse direction. Given a problem about sets, think of the universal set as being a set of logical possibilities and think of a subset as being the truth set of a statement. Hence we can "translate" a problem about sets into a problem about compound statements.

So far we have discussed only the truth sets assigned to compound statements involving $\vee$, $\wedge$, and $\sim$. All the other connectives can be defined in terms of these three basic ones, so that we can deduce what truth sets should be assigned to them. For example, we know that $p \rightarrow q$ is equivalent to $\sim p \vee q$. Hence the truth set of $p \rightarrow q$ is the same as the truth set of $\sim p \vee q$, that is, it is $\tilde{P} \cup Q$. The Venn diagram for $p \rightarrow q$ is shown in Figure 9, where the shaded area is the truth set for the statement. Observe that the unshaded area in Figure 9 is the set $P - Q = P \cap \tilde{Q}$, which is the truth set of the statement $p \wedge \sim q$. Thus the shaded area is the set $(\widetilde{P - Q}) = \widetilde{P \cap \tilde{Q}}$, which is the truth set of the statement $\sim[p \wedge \sim q]$. We have thus discovered the fact that $(p \rightarrow q)$, $(\sim p \vee q)$, and $\sim(p \wedge \sim q)$ are equivalent. It is always the case that two compound statements are equivalent if and only if they have the same truth sets. Thus we can test for equivalence by checking whether they have the same Venn diagram.

Suppose that p is a statement that is logically true. What is its truth set? Now p is logically true if and only if it is true in every logically possible case, so that the truth set of p must be $\mathcal{U}$. Similarly, if p is logically false, then it is false for every logically possible case, so that its truth set is the empty set $\mathcal{E}$.

Finally, let us consider the implication relation. Recall that p implies q if and only if the conditional $p \rightarrow q$ is logically true. But $p \rightarrow q$ is logically true if and only if its truth set is $\mathcal{U}$, that is, $\widetilde{(P - Q)} = \mathcal{U}$, or $(P - Q) = \mathcal{E}$. From Figure 4 we see that if $P - Q$ is empty, then P is contained in Q. We shall symbolize the containing relation as follows: $P \subset Q$ means "P is a subset of Q." We conclude that $p \Rightarrow q$ if and only if $P \subset Q$.

Figure 10 supplies a "dictionary" for translating from statement language to set language, and back. To each statement relative to a set of possibilities $\mathcal{U}$ there corresponds a subset of $\mathcal{U}$—namely, the truth set of the statement.

Statement Language	Set Language
r	R
s	S
$\sim r$	$\tilde{R}$
$r \vee s$	$R \cup S$
$r \wedge s$	$R \cap S$
$r \rightarrow s$	$\widetilde{(R - S)}$
$r \Rightarrow s$	$R \subset S$
$r \Leftrightarrow s$	$R = S$

Figure 10

This is shown in lines 1 and 2 of the figure. To each connective there corresponds an operation on sets, as illustrated in the next four lines. And to each relation between statements there corresponds a relation between sets, examples of which are shown in the last two lines of the figure.

EXAMPLE 1 Verify by means of a Venn diagram that the statement $[p \vee (\sim p \vee q)]$ is logically true. The assigned set of this statement is $[P \cup (\tilde{P} \cup Q)]$, and its Venn diagram is shown in Figure 11. In that figure the set P is shaded in color, and the set $\tilde{P} \cup Q$ is shaded in gray. Their union is the entire shaded area, which is $\mathcal{U}$, so that the compound statement is logically true.

EXAMPLE 2 Demonstrate by means of Venn diagrams that $p \vee (q \wedge r)$ is equivalent to $(p \vee q) \wedge (p \vee r)$. The truth set of $p \vee (q \wedge r)$ is the entire shaded area

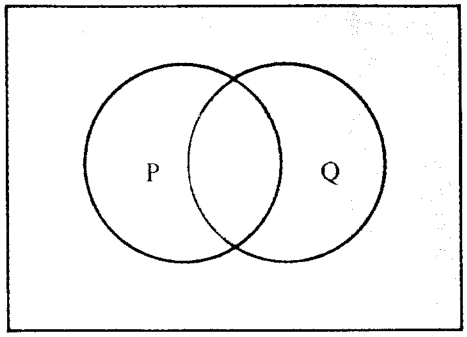


Figure 11

of Figure 12a, and the truth set of $(p \vee q) \wedge (p \vee r)$ is the area in Figure 12b shaded in both color and gray. Since these two sets are equal, we see that the two statements are equivalent.

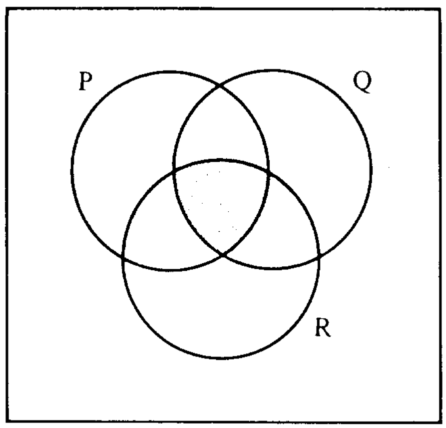


Figure 12a

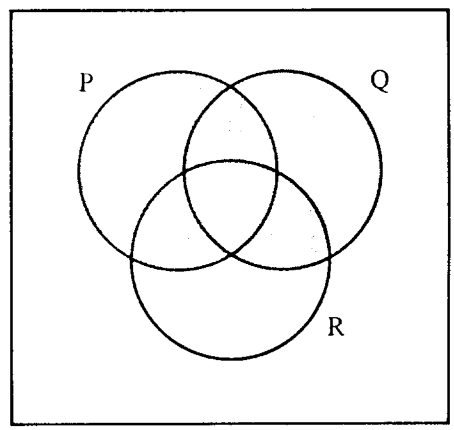


Figure 12b

EXAMPLE 3 Show by means of a Venn diagram that q implies $p \rightarrow q$. The truth set of $p \rightarrow q$ is the shaded area in Figure 9. Since this shaded area includes the set Q, we see that q implies $p \rightarrow q$.

EXERCISES

1. Use Venn diagrams to test the following statements for equivalences.

(a) $\sim(p \vee q)$.
(b) $\sim p \vee \sim q$.
(c) $\sim(p \wedge q)$.
(d) $\sim p \wedge \sim q$.
(e) $q \rightarrow p$.
(f) $\sim(\sim p \rightarrow q)$.

[*Ans.* (a), (d), and (f) are equivalent; (b) $\Leftrightarrow$ (c).]

2. Use Venn diagrams to tell which of the following statements are logically true and which are logically false.

(a) $p \wedge \sim p$.

(b) $(p \wedge q) \vee (\sim p \vee \sim q)$. [*Ans.* Logically true.]
(c) $(p \wedge q) \vee (p \wedge \sim q)$.
(d) $\sim p \vee (q \rightarrow p)$.
(e) $p \rightarrow (q \rightarrow p)$.
(f) $\sim(p \rightarrow q) \wedge q$.

3. Derive a test for inconsistency of p and q, using Venn diagrams.

4. Three or more statements are said to be inconsistent if they cannot all be true. What does this say about their truth sets?

5. Use Venn diagrams for the following statements to test whether one implies the other.
(a) $p \wedge q; p \wedge \sim q$. (b) $\sim(q \rightarrow p); p \rightarrow q$.
(c) $p \wedge q; \sim p \vee q$. (d) $\sim p \wedge q; q$.
(e) $p \vee q; p \rightarrow (\sim p \rightarrow q)$. (f) $(p \rightarrow q) \wedge \sim q; q \rightarrow p$.

6. Find statements having each of the following as truth sets.
(a) $(P \cap Q) - R$.
(b) $(R - Q) \cup (Q - R)$.
(c) $P - (\widetilde{Q \cup R})$.
(d) $(\widetilde{P \cap Q}) \cup (P \cup R)$.

7. Use truth tables to find whether the following sets are all different.
(a) $(P \cap Q \cap \tilde{R}) \cup (P \cap \tilde{Q} \cap R) \cup (\tilde{P} \cap Q \cap R)$.
(b) $[P - (Q \cup R)] \cup (R \cap Q)$.
(c) $Q \cap \tilde{R}$.
(d) $(P \cap Q \cap \tilde{R}) \cup (\tilde{P} \cap Q \cap \tilde{R})$.
(e) $[(P \cap Q) \cup (P \cap R) \cup (r \cap Q)] - (p \cap Q \cap R)$.
(f) $[(P \cap \tilde{Q} \cap \tilde{R}) \cup [\widetilde{(Q \cup R) - (Q \cap R)}] - (\tilde{P} \cap \tilde{Q} \cap \tilde{R})$.

8. Use truth tables to find whether each of the following sets is empty.
(a) $(P - Q) \cap (Q - P)$. [*Ans.* Empty.]
(b) $(\tilde{P} \cup Q) \cap (\tilde{Q} \cup R) \cap (\widetilde{\tilde{P} \cup R})$.
(c) $(\widetilde{P \cap R}) \cap (\tilde{P} \cap \tilde{Q})$. [*Ans.* Not empty.]
(d) $(\widetilde{P \cup R}) \cap \tilde{Q}$.
(e) $(P \cap Q) - P$.
(f) $(P \cap (Q - R)) - ((P \cap Q) - R)$.

9. Show, both by the use of truth tables and by the use of Venn diagrams, that $p \vee (q \wedge r)$ is equivalent to $(p \vee q) \wedge (p \vee r)$.

10. Use truth tables for the following pairs of sets to test whether one is a subset of the other.
(a) $P \cap Q; \quad [R - (\widetilde{P \cup Q})]$.
(b) $(\tilde{P} \cap Q) \cap (\tilde{Q} \cup R); \tilde{P} \cup R$.
(c) $P \cap (Q \cup R); \quad P \cap Q$.
(d) $P \cap \tilde{Q}; \quad \tilde{P} \cap Q$.
(e) $Q; \quad (\tilde{P} \cup Q) \cap P$.
(f) $P - (Q - R); (P - Q) - R$.

11. The *symmetric difference* of P and Q is defined to be $(P - Q) \cup (Q - P)$. What connective corresponds to this set operation?

4 PERMUTATIONS

The first step in the analysis of a scientific problem is the determination of the set of logical possibilities. Next it is often necessary to determine how many different possible outcomes there are. We shall find this particularly important in probability theory. Hence it is desirable to develop general techniques for solving counting problems. In this section and the next we shall discuss the two most important cases in which it is possible to achieve formulas that solve the problem. When a formula cannot be derived, one must resort to certain other general counting techniques, tricks, or, in the last resort, complete enumeration of the possibilities.

As a first problem let us consider the number of ways in which a set of n different objects can be arranged. A *listing* of n different objects *in a certain order* is called a *permutation* of the n objects. We consider first the case of three objects, a, b, and c. We can exhibit all possible permutations of these three objects as paths of a tree, as shown in Figure 13. Each path

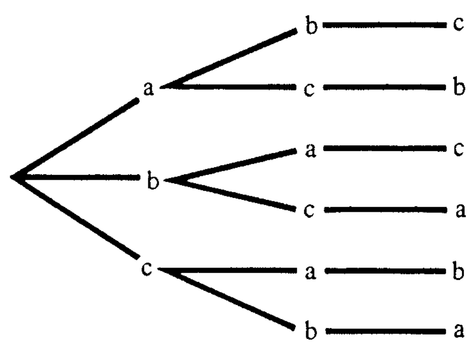


Figure 13

exhibits a possible permutation, and there are six such paths. We know there are six paths from the following argument: we have 3 choices for the first object; after this first choice we can choose the second object in 2 ways; then the last object must be listed; thus the total number of listings is $3 \cdot 2 \cdot 1 = 6$. We could also list these permutations as follows:

abc,	bca,
acb,	cab,
bac,	cba.

If we were to construct a similar tree for n objects, we would find that the number of paths could be found by multiplying together the numbers $n, n - 1, n - 2$, continuing down to the number 1. The number obtained in this way occurs so often that we give it a symbol, namely $n!$, which is read "n factorial." Thus, for example, $3! = 3 \cdot 2 \cdot 1 = 6$, $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$, and so on. For reasons that will be clear later, we define $0! = 1$. Thus we can say *there are n! different permutations of n distinct objects.*

EXAMPLE 1 Seven different machining operations are to be performed on a part, but they may be performed in any sequence. We may then consider $7! = 5040$ different orders in which the operations may be performed.

EXAMPLE 2 Ten workers are to be assigned to 10 different jobs. In how many ways can the assignments be made? The first worker may be assigned in 10 possible ways, the second in any of the 9 remaining ways, the third in 8, and so forth: there are $10! = 3{,}628{,}800$ possible ways of assigning the workers to the jobs.

EXAMPLE 3 A company has n directors. In how many ways can they be seated around a circular table at a board meeting, if two arrangements are considered different only if at least one person has a different person sitting on his right in the two arrangements? To solve the problem, consider one director in a fixed position. There are $(n - 1)!$ ways in which the other people may be seated. We have now counted all the arrangements we wish to consider different. Thus there are also $(n - 1)!$ possible seating arrangements.

For many counting problems it is not possible to give a simple formula for the number of possible cases. In many of these the only way to find the number of cases is to draw a tree and count them. In some problems, the following general principle is useful.

A General Principle If one thing can be done in exactly r different ways, for each of these a second thing can be done in exactly s different ways, for each of the first two, a third can be done in exactly t ways, and so on, then the sequence of things can be done in $r \cdot s \cdot t \ldots$ ways.

EXAMPLE 4 Suppose we live in town X and want to go to town Z by passing through town Y. If there are three roads from X to Y, and two roads from Y to Z, in how many ways can we go from town X to town Z? By applying the general principle we see that there are $3 \cdot 2 = 6$ ways.

The validity of this general principle can be established by thinking of a tree representing all the ways in which the sequence of things can be done. There would be r branches from the starting position. From the ends of each of these r branches there would be s new branches, and from each of these t new branches, and so on. The number of paths through the tree would be given by the product $r \cdot s \cdot t \ldots$.

EXAMPLE 5 The number of permutations of n distinct objects is a special case of this principle. If we were to list all the possible permutations, there would be n possibilities for the first, for each of these $n - 1$ for the second, etc., until we came to the last object, and for which there is only one possibility. Thus there are $n(n - 1) \ldots 1 = n!$ possibilities in all.

EXAMPLE 6 An automobile manufacturer produces four different models; models A and B can come in any of four body styles—sedan, hardtop, convertible, and station wagon—while models C and D come only as sedans or hardtops. Each can can come in one of nine colors. Thus models A and B each have $4 \cdot 9 = 36$ distinguishable types, while C and D have $2 \cdot 9 = 18$ types, so that in all

$$2 \cdot 36 + 2 \cdot 18 = 108$$

different car types are produced by the manufacturer.

EXAMPLE 7 Suppose there are n applicants for a certain job. Three interviewers are asked independently to rank the applicants according to their suitability. It is decided that an applicant will be hired if he is ranked first by at least two of the three interviewers. What fraction of the possible reports would lead to the acceptance of some candidate? We shall solve this problem by finding the fraction of the reports that do not lead to an acceptance and subtract this answer from 1. Frequently an indirect attack of this kind is easier than the direct approach. The total number of reports possible is $(n!)^3$, since each interviewer can rank the men in $n!$ different ways. If a particular report does not lead to the acceptance of a candidate, it must be true that each interviewer has put a different man in first place. By our general principle, this can be done in $n(n - 1)(n - 2)$ different ways. For each possible first choice, there are $[(n - 1)!]^3$ ways in which the remaining men can be ranked by the interviewers. Thus the number of reports that do not lead to acceptance is

$$n(n - 1)(n - 2)[(n - 1)!]^3.$$

Dividing this number by $(n!)^3$, we obtain

$$\frac{(n - 1)(n - 2)}{n^2}$$

as the fraction of reports that fail to accept a candidate. The fraction that leads to acceptance is found by subtracting this fraction from 1, which gives

$$\frac{3n - 2}{n^2}.$$

For the case of three applicants, we see that $\frac{7}{9}$ of the possibilities lead to acceptance. Here the procedure might be criticized on the grounds that even if the interviewers are completely ineffective and are essentially guessing, there is a good chance that a candidate will be accepted on the basis of the reports. For n equal to ten, the fraction of acceptances is only .28, so that it is possible to attach more significance to the interviewers' ratings, if they reach a decision.

EXERCISES

1. A salesman is going to call on five customers. In how many different sequences can he do this if he
   (a) Calls on all five in one day?
   (b) Calls on three one day and two the next?
   [*Ans.* (a) 120; (b) 120.]
2. A machine shop has three milling machines, five lathes, six drill presses, and three grinders. In how many ways can a part be routed that must first be ground, then milled, then turned on a lathe, and then drilled? In how many ways can it be routed if these four operations can be performed in any order?
3. A department store wants to classify each of its customers having a charge account by using a three-character code consisting of n letters followed by $3 - n$ digits. How large must n be if there are 5000 charge accounts? What if there are 10,000? 20,000?
4. Modify Example 7 so that, to be accepted, an applicant must be first in two of the interviewers' ratings and must be either first or second in the third interviewer's rating. What fraction of the possible reports lead to acceptance in the case of three applicants? In the case of n?
   [*Ans.* $\frac{4}{9}$; $4/n^2$.]
5. A company has six officers and six directors; two of the directors are officers. List the possible memberships of a committee of four men who are either officers or directors in terms of the number of members who are (a) just officers, (b) just directors, and (c) both officers and directors.
6. In Exercise 5, how many ways are there of obtaining a committee of four consisting of
   (a) Three who are just officers and one who is officer and director?
   (b) One who is just an officer, one who is just a director, and two who are officers and directors?
   (c) At least two who are only directors and at least one who is officer and director?
   (d) At least two officers and at least two directors (assuming a man who is both officer and director satisfies both quotas)?
   [*Ans.* 160.]
7. Show the possible arrangement of machines A, B, C, and D in a circle. How many are there?
8. How many possible ways are there of seating six people A, B, C, D, E, and F at a circular table if
   (a) A must always have B on his right and C on his left?

(b) A must always sit next to B?

(c) A cannot sit next to B?

9. In seating n people around a circular table, suppose we distinguish between two arrangements only if at least one person has at least one different person sitting next to him in the two arrangements. That is, we do not regard two arrangements as different simply because the right-hand and left-hand neighbors of a person have interchanged places. Now how many distinguishable arrangements are there?

10. A certain symphony orchestra always plays one of the 41 Mozart symphonies, followed by one of 25 different modern works, followed by one of the 9 Beethoven symphonies.

(a) How many different programs can it play?

(b) How many different programs can be given if the pieces can be played in any order?

(c) How many three-piece programs are possible if more than one piece from the same category can be played?

11. Find the number of arrangements of the five symbols that can be distinguished. (The same letters with different subscripts indicate distinguishable objects.)

(a) A_1, A_2, B_1, B_2, B_3. [*Ans.* 120.]

(b) A, A, B_1, B_2, B_3. [*Ans.* 60.]

(c) A, A, B, B, B. [*Ans.* 10.]

12. Show that the number of distinguishable arrangements possible for n objects, n_1 of type 1, n_2 of type 2, and so on for r different types is

$$\frac{n!}{n_1!n_2! \cdots n_r!}.$$

13. A student takes a five-question multiple-choice test, each question having answer a, b, c, or d. If he knows that the answers to the test consist of two a's and one each of b, c, and d and he answers accordingly, in how many different ways can be answer the test? In what fraction of these will he get four or more right answers? In what fraction will he get three or more right? [*Partial Ans.* 60.]

14. How many signals can a ship show if it has eight flags and a signal consists of five flags hoisted vertically on a rope? [*Ans.* 6720.]

15. We must arrange four green, one red, and four blue books on a single shelf. All books are distinguishable.

(a) In how many ways can this be done if there are no restrictions?

(b) In how many ways if books of the same color must be grouped together?

(c) In how many ways if, in addition to the restriction in (b) the red books must be to the left of the blue books?

(d) In how many ways if, in addition to the restrictions in (b) and (c), the red and blue books must not be next to each other?

[*Ans.* 576.]

16. **(a)** How many five-digit numbers can be formed from the digits 1, 2, 3, 4, 5 using each digit only once?
(b) How many of these numbers are less than 33,000?

17. A housewife who has just returned from shopping realizes that she has left her sunglasses at either the bank, the post office, the drugstore, or the grocery store, and so she must go back and search for them. Assume that when she returns to the building where she left them, she finds them and then goes directly home.
(a) In how many different orders can all four places be searched?
(b) Assume we now know that she found her glasses at the third place she returned to. How many different searches can she have made?
(c) If we know only that her glasses were left at the bank, how many different searches can she have made?

5 LABELING PROBLEMS

The second general type of counting problem that we want to consider may be described as follows. We have n objects and we wish to label each of these objects with one of r different types of labels. To be more specific, we wish to determine the total number of ways that we can label the n objects with r labels if n_1 of the objects are to be given the first type of label, n_2 the second type, and so on, where $n_1, n_2, \ldots, n_r$ are given nonnegative integers such that $n_1 + n_2 + \cdots + n_r = n$.

As an example assume that we have eight customers, A, B, C, D, E, F, G, and H, and we wish to assign to each of them one of three salesmen, Brown, Jones, or Smith. And we want to make this assignment so that Brown is assigned to three customers, Jones to three, and Smith to two. Notice that we can interpret the problem as that of assigning a label—Brown, Jones, or Smith—to each of the eight customers. In how many ways can this assignment be made?

One way to assign the customers is to list them in some arbitrary order (that is, select a permutation of them) and then assign Brown to the first three, Jones to the next three, and Smith to the last two. There are 8! permutations or listings of the customers, but not all of these lead to different assignments. For instance, consider the following assignment:

$$|\text{BCA}|\text{DFE}|\text{HG}|.$$

Here, Brown is assigned to B, C, and A, Jones to D, F, and E, and Smith to H and G. Notice that another permutation such as

$$|\text{ABC}|\text{DEF}|\text{GH}|$$

gives the same customer assignments, since it differs only in the sequences for particular salesmen. There are $3! \cdot 3! \cdot 2!$ such listings, since we can arrange the three customers of Brown in 3! different ways, and for each of these, the customers of Jones in 3! different ways, and for each of these, the customers of Smith in 2! different ways. Since there are $3! \cdot 3! \cdot 2!$

different listings that lead to the same assignments and 8! listings in all, there are $8!/(3!\cdot 3!\cdot 2!)$ different assignments of customers to salesmen.

The same argument could be carried out for r salesmen and n customers with n_1 assigned to the first salesman, n_2 to the second, and so on. In fact there is really nothing special about the argument for this example, so we have the following basic result. Let $n_1, n_2, \ldots, n_r$ be nonnegative integers with $n_1 + n_2 + \cdots + n_r = n$. Then:

The number of ways that n objects can be labeled with r different types of labels, n_1 with the first type, n_2 with the second, and so on, is

$$\frac{n!}{n_1!n_2!\cdots n_r!}$$

We shall denote this number by the symbol

$$\binom{n}{n_1,n_2,\ldots,n_r}.$$

The special case when $r = 2$, meaning that there are just two types of labels, is particularly important. The problem is often stated in the following way. We are given a set of n elements; in how many ways can we choose a subset with j elements? If we interpret the problem to mean labeling each element as either "in the set" or "not in the set," we see that it is just a labeling problem whose answer is

$$\binom{n}{j,\, n-j} = \frac{n!}{j!(n-j)!};$$

and hence this is also the number of subsets with j elements. The notation $\binom{n}{j,\, n-j}$ is commonly shortened to $\binom{n}{j}$. These numbers are known as *binomial coefficients*.

Notice that every time we choose a subset of j elements to put in our subset we are also choosing a subset of $n - j$ elements to leave out. In this way we see that

$$\binom{n}{j} = \binom{n}{j,\, n-j} = \binom{n}{n-j}.$$

EXAMPLE 1 The aces and kings are removed from a bridge deck, and from the resulting eight-card deck a hand of two cards is dealt. How many such two-card hands are there? By the principle just stated we see that there are $\binom{8}{2} = \binom{8}{6} = 28$ such hands, since choosing a two-card hand is just the same as choosing the remaining six cards to keep in the deck. (The reader should enumerate the 28 possible two-card hands.)

EXAMPLE 2 A company buys a certain electronic component from three vendors. In how many ways can it place six orders, two with vendor A, three with vendor B, and one with vendor C? This is just the problem of labeling each of the six orders with one of three labels, A, B, or C. There are

$$\binom{6}{2, 3, 1} = \frac{6!}{2!3!1!} = 60$$

ways of carrying out the labeling.

EXAMPLE 3 On August 20, 1970, 1551 different stock issues were traded on the New York Stock Exchange. Of these, 701 advanced, 530 declined, and 320 closed unchanged from the previous day. In how many ways could this have happened? We must label each stock as "advanced," "declined," or "unchanged." There are

$$\frac{1551!}{701!530!320!}$$

different ways in which this particular result could occur. This number is approximately equal to $1.1 \cdot 10^{705}$.

EXAMPLE 4 This example will be important in probability theory, which we take up in the next chapter. If a coin is tossed six times, there are 2^6 possibilities for the outcome of the six throws, since each throw can result in either a head or a tail. How many of these possibilities result in four heads and two tails? We can interpret each assignment of outcomes to be a labeling of each integer from 1 to 6 with either H or T, corresponding to whether heads or tails came up on that toss. Since we required that four be labeled H and two T, the answer is $\binom{6}{4} = 15$. For n throws of a coin, a similar analysis shows that there are $\binom{n}{r}$ different sequences of H's and T's of length n that have exactly r heads and $n - r$ tails.

EXERCISES

1. Compute the following numbers:

(a) $\binom{8}{6}$. [*Ans.* 28.]

(b) $\binom{4}{2}$.

(c) $\binom{2}{1}$.

(d) $\binom{780}{779}$. [*Ans.* 780.]

(e) $\binom{10}{0}$.

(f) $\binom{3}{2,0,1}$.

(g) $\binom{5}{2,2,1}$. [*Ans.* 30.]

(h) $\binom{8}{4,1,3}$.

2. Show that

$$\binom{a}{b} = \frac{a \cdot (a-1) \cdot (a-2) \cdot \cdots \cdot (a-b+2) \cdot (a-b+1)}{b \cdot (b-1) \cdot (b-2) \cdot \cdots \cdot 2 \cdot 1},$$

where there are exactly b terms in both the numerator and the denominator.

3. A group of six workers is to be assigned to six of nine available jobs. If we are only interested in which jobs are assigned, and not the specific worker-job assignments and if all of the workers are assigned jobs, in how many ways can the jobs be assigned to the workers? How many possibilities are there for the unassigned jobs, if three of the jobs are sure to be assigned? [*Ans.* 84, 20.]

4. Give an interpretation for $\binom{n}{0}$ and aso for $\binom{n}{n}$. Can you now give a reason for making $0! = 1$?

5. A hospital has just received eight chairs, four red and four blue. In how many different ways can these be distributed between two waiting rooms if each room must receive at least three chairs and at least one chair of each color? (Assume chairs of the same color are of different types, and thus distinguishable.)

6. From a lot containing six pieces, three good and three defective, a sample of three pieces is drawn. If we distinguish each piece, find the number of possible samples that can be formed

(a) With no restrictions. [*Ans.* 20.]
(b) With three good pieces and no defectives. [*Ans.* 1.]
(c) With two good pieces and one defective. [*Ans.* 9.]
(d) With one good piece and two defectives. [*Ans.* 9.]
(e) With no good pieces and three defectives. [*Ans.* 1.]

What is the relation between your answer in part (a) and the answers to the remaining four parts?

7. Exercise 6 suggests that the following should be true:

$$\binom{2n}{n} = \binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \binom{n}{2}\binom{n}{n-2} + \cdots + \binom{n}{n}\binom{n}{0} = \binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2.$$

Show that it is true.

8. Consider a town with four plumbers, A, B, C, and D. On a certain day eight residents of the town telephone for a plumber. If each resident selects a plumber from the telephone directory, in how many ways can it happen that
   (a) Three residents call A, three call B, one calls C, and one calls D?
   (b) The distribution of calls to the plumbers is three, three, one, and one? [*Ans.* 6720.]

9. In a class of 20 students, grades of A, B, C, D, and F are to be assigned. Omit arithmetic details in answering the following:
   (a) In how many ways can this be done if there are no restrictions?
   (b) In how many ways can this be done if the grades are assigned as follows: 2 A's, 3 B's, 10 C's, 3 D's, and 2 F's?
   (c) In how many ways can this be done if the following rules are to be satisfied: exactly 10 C's; the same number of A's as F's; the same number of B's and D's; always more B's than A's?

$$\left[\textit{Ans. } \binom{20}{5, 10, 5} + \binom{20}{1, 4, 10, 4, 1} + \binom{20}{2, 3, 10, 3, 2}.\right]$$

10. In how many ways can a machine produce nine pieces, five of which are good and four of which are defective? In how many ways if no two consecutive pieces are both good or both defective?

11. Establish the identity

$$\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}$$

for $n \geq r \geq k$ in two ways, as follows:
   (a) Replace each expression by a ratio of factorials and show that the two sides are equal.
   (b) Consider the following problem: From a set of n people a committee or r is to be chosen, and from these r people a steering subcommittee of k people is to be selected. Show that the two sides of the identity give two different ways of counting the possibilities for this problem.

12. A brewing company contracts with a television station to show three spot commercials a week for 52 weeks. The commercials consist of a series of cartoons. It is decided that in no two weeks will exactly the same three cartoons be shown. What is the minimum number of cartoons that will accomplish this?

13. Twenty bridge players enter a tournament and form ten partnerships. Seven of the players are good bridge players, ten are mediocre, and three are terrible. How many possibilities are there for the winning partnership if we know that the winning partnership
    (a) Contained no terrible player? [*Ans.* 136.]
    (b) Contained two good players?
    (c) Contained one good and one mediocre player?
14. Referring to Exercise 13, answer the following questions, omitting arithmetic computations.
    (a) How many possible sets of ten partnerships are there?
    (b) How many sets of ten partnerships are possible if no two terrible players play together?
    (c) How many sets of ten partnerships are possible if, in addition to restriction (b), no two good players play together and no two mediocre players play together?
15. A group of nine people is to be divided into three committees of two, three, and six members, respectively. The chairman of the group is to serve on all three committees and is the only member of the group who serves on more than one committee. In how many ways can the committee assignments be made? [*Ans.* 168.]
16. A landlord decides to repaint two of his apartments, each having five rooms. Assuming that he uses only green, yellow, and blue paint and that each room is to be painted with only one color.
    (a) How many different ways are there of painting the apartments? [*Ans.* 3^{10}.]
    (b) How many different ways are there of painting the apartments, given that no more than two colors are to be used in any one apartment? [*Ans.* 8649.]

6 SOME PROPERTIES OF BINOMIAL COEFFICIENTS

The binomial coefficients $\binom{n}{j}$ introduced in Section 5 will play an important role in our future work. We give here some of the more important properties of these numbers.

A convenient way to obtain these numbers is given by the famous *Pascal triangle,* shown in Figure 14. To obtain the triangle we first write the 1's down the sides. Any of the other numbers in the triangle has the property that it is the sum of the two adjacent numbers in the row just above. Thus the next row in the triangle is 1, 6, 15, 20, 15, 6, 1. To find the binomial coefficient $\binom{n}{j}$ we look in the row corresponding to the number n and see where the diagonal line corresponding to the value of j intersects this row. For example, $\binom{4}{2} = 6$ is in the row marked $n = 4$ and on the diagonal marked $j = 2$.

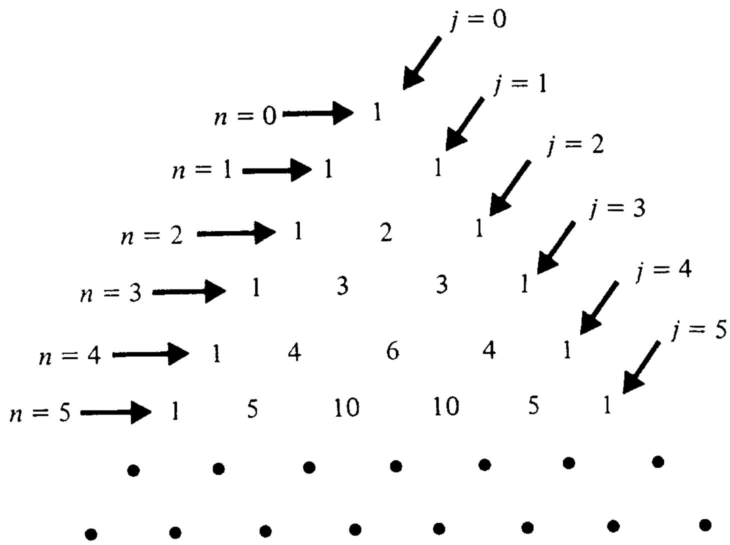


Figure 14

The property of the binomial coefficients upon which the triangle is based is

$$\binom{n+1}{j} = \binom{n}{j-1} + \binom{n}{j}.$$

This fact can be verified directly (see Exercise 5), but the following argument is interesting in itself. The number $\binom{n+1}{j}$ is the number of subsets with j elements that can be formed from a set of $n + 1$ elements. Select one of the $n + 1$ elements, x. Of the $\binom{n+1}{j}$ subsets some contain x, and some do not. The latter are subsets of j elements formed from n objects, and hence there are $\binom{n}{j}$ such subsets. The former are constructed by adding x to a subset of $j - 1$ elements formed from n elements, and hence there are $\binom{n}{j-1}$ of them. Thus

$$\binom{n+1}{j} = \binom{n}{j-1} + \binom{n}{j}.$$

If we look again at the Pascal triangle, we observe that the numbers in a given row increase for a while, and then decrease. In fact, they increase to a unique maximum when n is even or to two equal maxima when n is odd.

An important application of binomial coefficients is in the expansion of products of the form $(x + y)^3$, $(a - 2b)^{10}$, and so on. We shall derive a general formula for these by making use of the binomial coefficients.

Consider first the special case $(x + y)^3$. We write this as

$$(x + y)^3 = (x + y)(x + y)(x + y).$$

To perform the multiplication, we choose either an x or y from each of the three factors and multiply our choices together; we do this for all possible choices and add the results. To state this as a labeling problem, note that we want to label each of the three factors with the two labels x and y. In how many ways can we do this using two x labels and one y? The preceding section gives the answer $\binom{3}{2} = 3$. Hence the coefficient of x^2y in the expansion of the binomial is 3. More generally, the coefficient of the term of the form $x^j y^{3-j}$ will be $\binom{3}{j}$ for $j = 0, 1, 2, 3$. Thus we can write the desired expansion as

$$\begin{aligned}(x + y)^3 &= \binom{3}{3}x^3 + \binom{3}{2}x^2y + \binom{3}{1}xy^2 + \binom{3}{0}y^3 \\ &= x^3 + 3x^2y + 3xy^2 + y^3.\end{aligned}$$

Binomial Theorem The expansion of $(x + y)^n$ is given by

$$(x + y)^n = \binom{n}{n}x^n + \binom{n}{n-1}x^{n-1}y + \binom{n}{n-2}x^{n-2}y^2 + \cdots + \binom{n}{1}xy^{n-1} + \binom{n}{0}y^n.$$

EXAMPLE 1 Let us find the expansion for $(a - 2b)^3$. To fit this into the binomial theorem, we think of x as being a and y as being $-2b$. Then we have

$$\begin{aligned}(a - 2b)^3 &= a^3 + 3a^2(-2b) + 3a(-2b)^2 + (-2b)^3 \\ &= a^3 - 6a^2b + 12ab^2 - 8b^3.\end{aligned}$$

EXERCISES

1. Extend the Pascal triangle to $n = 16$. Save the result for later use.
2. (a) Show that a set with n elements has 2^n subsets. [*Hint:* Assume you have two different kinds of labels: "in the subset" and "not in the subset." In how many different ways can we label the n elements of the set?]

   (b) Prove that

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n,$$

   using the fact that a set with n elements has 2^n subsets.
3. Using the fact that

$$\binom{n}{j+1} = \frac{n-j}{j+1}\binom{n}{j},$$

compute $\binom{27}{s}$ for $s = 1, 2, 3, 4, 5$ starting with the fact that $\binom{27}{0} = 1$.

4. For $n \leq m$ prove that

$$\binom{m}{0}\binom{n}{0} + \binom{m}{1}\binom{n}{1} + \binom{m}{2}\binom{n}{2} + \cdots + \binom{m}{n}\binom{n}{n} = \binom{m+n}{n}$$

by carrying out the following two steps:

(a) Show that the left-hand side counts the number of ways of choosing equal numbers of men and women from sets of m men and n women.

(b) Show that the right-hand side also counts the same number by showing that we can select equal numbers of men and women by selecting any subset of n persons from the whole set, and then combining the men selected with the women not selected.

5. Prove that

$$\binom{n+1}{j} = \binom{n}{j-1} + \binom{n}{j},$$

using only the fact that

$$\binom{n}{j} = \frac{n!}{j!\,(n-j)!}.$$

6. Expand by the binomial theorem:

(a) $(x + 1)^4$. [*Ans.* $x^4 + 4x^3 + 6x^2 + 4x + 1$.]

(b) $(2x + y)^3$.

(c) $(x - 2)^5$.

(d) $(2a - x)^4$.

(e) $(3x + 4y)^3$.

(f) $(100 - 2)^4$.

7. Using the binomial theorem, prove that

(a) $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$.

(b) $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots \pm \binom{n}{n} = 0$ for $n > 0$.

*7 APPLICATIONS OF COUNTING TECHNIQUES

One of the important areas in which finite mathematics is applied is in solving *combinatorial decision problems.* In such problems there are a finite number of ways in which a certain procedure can be carried out, and for each of these ways a cost or value can be calculated. We want to select a way of carrying out the procedure that has minimum cost or maximum value.

One method for solving combinatorial decision problems is to enumerate all the possible ways of carrying out the procedure and selecting the one that is most desirable. Although this is theoretically possible, it may be practically impossible since the number of alternatives frequently is too large to enumerate completely even with the aid of an electronic computer. Hence methods that do not require complete enumeration are needed to solve such problems.

We illustrate the use of counting techniques to help solve such problems.

EXAMPLE 1 Consider a city with a grid of streets as shown in Figure 15. Jones and Smith are at corner A and want to go to corner B, which is four blocks east and five blocks north of A. In how many ways can they make the journey and travel exactly nine blocks?

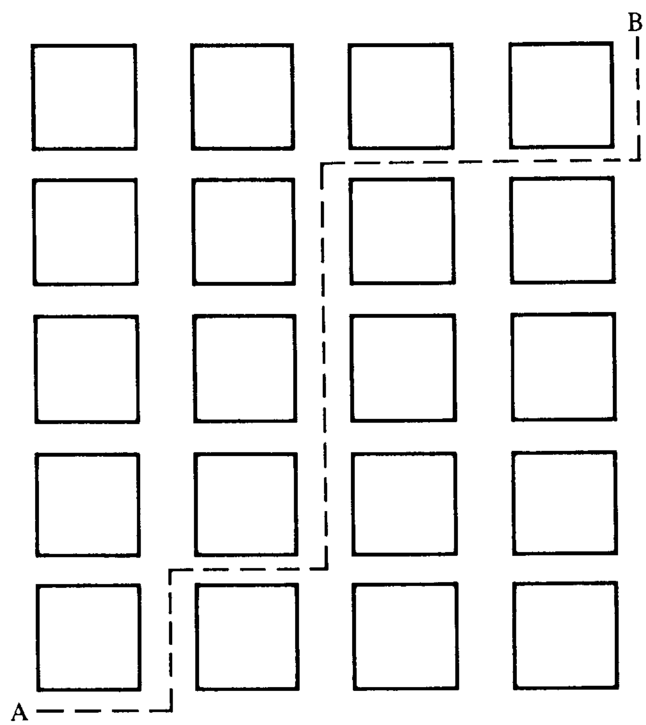


Figure 15

You may wish to try to count all possible ways, but if you try you are very likely to become tired and confused. This would be especially true if the distances were larger, say 100 blocks east and 100 blocks north! However we can reformulate the problem so that it is easy if we notice that all that Jones and Smith have to do is to make nine decisions, each decision being to go a block either east or north, with exactly four of the nine decisions being to go east and the remaining five to go north. For instance, one series of decisions, leading to the path shown dotted in Figure 15, is represented by the decisions

east, north, east, north, north, north, east, east, north.

Once we understand this reformulation of the problem, its solution is easy, since the number of ways we can choose four out of nine decisions to be east (or equally well five out of nine to be north) is clearly

$$\binom{9}{4} = \binom{9}{5} = \frac{6 \cdot 7 \cdot 8 \cdot 9}{1 \cdot 2 \cdot 3 \cdot 4} = 126 \text{ paths.}$$

The general problem is just as easy. If Jones and Smith are going h blocks east and k blocks north, the total number N of possible paths is given by

$$N = \binom{h+k}{h} = \binom{h+k}{k}.$$

Let us make this into a combinatorial decision problem by requiring that the number of corners turned on the path be a minimum. At least one corner must be turned. A little experimentation will show that two paths exist which turn at only one corner. These are (1) go four blocks east and five blocks north and (2) go five blocks north and four blocks east. These two answers solve the decision problem.

EXAMPLE 2 Suppose that point B is now three blocks east and five blocks north, and that the streets are alternate one-way east-west and north-south as indicated by the arrows in Figure 16. Smith is going to walk from A to B but Jones is going to take a taxi. We know that Smith must walk eight blocks and there are $\binom{8}{3} = 56$ possible paths he can take. After they arrive at B Jones and Smith compare notes. Smith said the taxi drove him ten blocks. Was the taxi driver honest?

The answer is yes, and it follows from the next theorem.

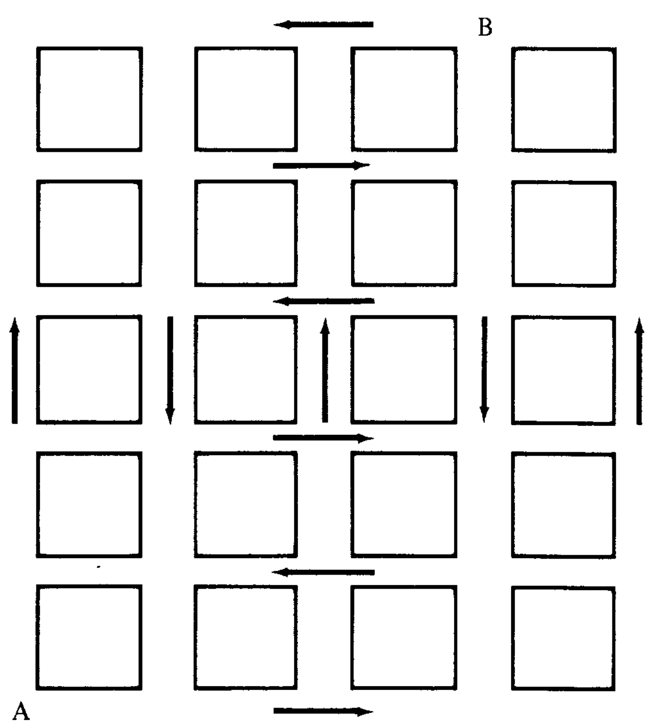


Figure 16

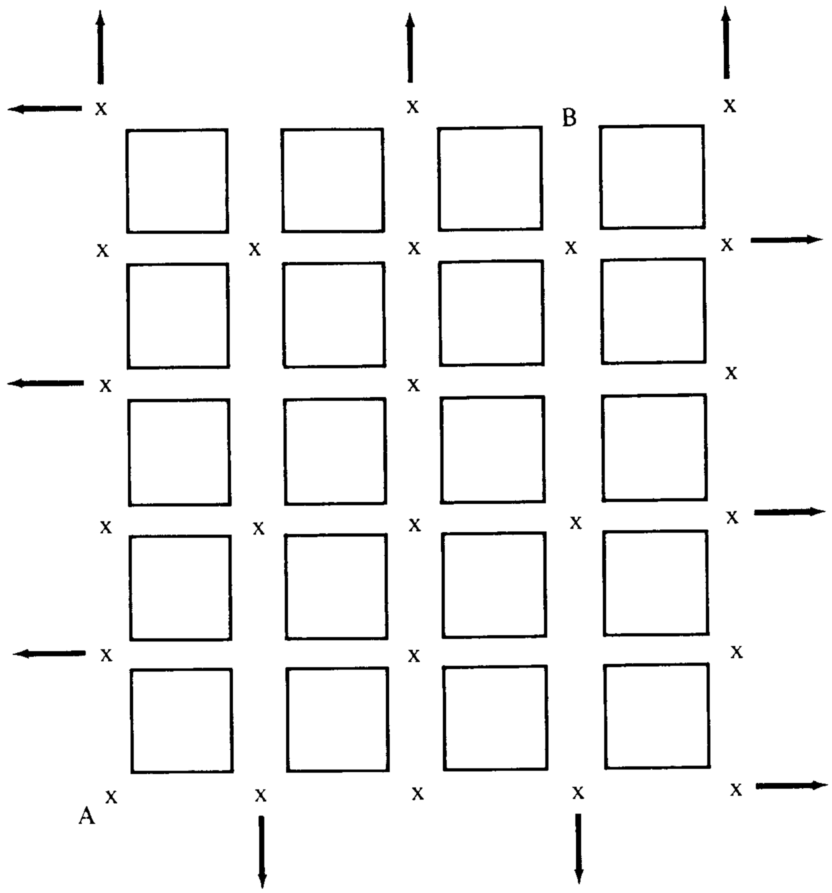


Figure 17

Theorem Consider a city with alternating one-way streets in both east-west and north-south directions. The shortest driving path between any two points A and B where B is northeast of A and the roads at A go east and north is either the same or exactly two blocks longer than the shortest walking path.

The proof is quite simple since we can easily show that, starting from A we can go to every corner of a four-block square except the center in exactly the same distance either by driving or walking (see the corners marked X in Figure 17.) To drive to the center of a four-block square (that is, to one of the corners not marked with an X in Figure 17) we drive first to an adjacent corner and then go to the center of the four-block square using the one-way streets. The latter step adds two additional blocks to the trip.

You may also wish to prove that if the roads at A go east and north and B is southwest of A, then the shortest driving path is either two or four blocks longer than the shortest walking path.

It is also true that the one-way-street pattern reduces the number of possible driving paths from A to B. In Exercise 1 you will be asked to show that there are six driving paths from A to B in Figure 16. Of these six there is only one that requires only two corners—and all the rest have more—so that the combinatorial decision problem now has a unique answer.

It often happens that a counting problem can be formulated in a number of different ways that sound quite different but that are in fact equivalent. And in one of these ways the answer may suggest itself readily. To illustrate how a reformulation can make a hard-sounding problem seem fairly easy, consider the following problem. Count the number of ways that n indistinguishable objects can be put into r cells. For instance, if there are three objects and three cells, the number of different ways can be enumerated as follows (using O for object and bars to indicate the sides of the cells):

OOO		
OO	O	
OO		O
O	OO	
O	O	O
O		OO
	OOO	
	OO	O
	O	OO
		OOO

We see that in this case there are ten ways the task can be accomplished. But the answer for the general case is not clear.

If we look at the problem in a slightly different manner, the answer suggests itself. Instead of putting the objects *in* the cells, we imagine putting the cells *around* the objects. In the above case we see that three cells are constructed from four bars. Two of these bars must be placed at the ends. We think of the two other bars together with our three objects as occupying five intermediate positions. Of these five intermediate positions we must choose two of them for bars and three for the objects. Hence the total number of ways we can accomplish the task is $\binom{5}{2} = \binom{5}{3} = 10$, which is the answer we got by counting all the ways.

For the general case we can argue in the same manner. We have r cells and n objects. We need $r + 1$ bars to form the r cells, but two of these must be fixed on the ends. The remaining $r - 1$ bars together with the n objects occupy $r - 1 + n$ intermediate positions. And we must choose $r - 1$ of these for the bars and the remaining n for the objects. Hence our task can be accomplished in

$$\binom{n + r - 1}{r - 1} = \binom{n + r - 1}{n}$$

different ways.

EXAMPLE 3 Seven people enter an elevator that will stop at five floors. In how many different ways can the people leave the elevator if we are interested only in the number that depart at each floor, and do not distinguish among the people? According to our general formula, the answer is

$$\binom{7+5-1}{7} = \binom{11}{7} = 330.$$

Suppose we are interested in finding the number of such possibilities in which at least one person gets off at each floor. We can then arbitrarily assign one person to get off at each floor, and the remaining two can get off at any floor. They can get off the elevator in

$$\binom{2+5-1}{2} = \binom{6}{2} = 15$$

different ways.

EXERCISES

1. In Figure 16 show that there are exactly six different driving paths from A to B.
2. Find the unique path from A to B requiring only two corners in Figure 16.
3. In Figure 15 suppose that point C is two blocks east and three blocks north of point A. How many ways are there of going from A to C and then to B by paths that are nine blocks long? [*Ans.* 60.]
4. In Figure 16 suppose that point C is one block east and one block north of A. How many driving paths are there for going from A to C and then to B that use the fewest number of blocks?
5. Four partners in a game require a total score of exactly 20 points to win. In how many ways can they accomplish this? $\left[Ans.\ \binom{23}{3}.\right]$
6. In how many ways can eight apples be distributed among four boys? In how many ways can this be done if each boy is to get at least one apple?
7. Suppose we have n balls and r boxes with $n \geq r$. Show that the number of different ways that the balls can be put into the boxes which insures that there is at least one ball in every box is $\binom{n-1}{r-1}$.
8. Identical prizes are to be distributed among five boys. It is observed that there are 15 ways that this can be done if each boy is to get at least one prize. How many prizes are there? [*Ans.* 7.]
9. By an ordered partition of n with r elements we mean a sequence of nonnegative integers, possibly some 0, written in a definite order, and having n as their sum. For instance, $\{1, 0, 3\}$ and $\{3, 0, 1\}$ are two

different ordered partitions of 4 with three elements. Show that the number of ordered partitions of n with r elements is $\binom{n+r-1}{n}$.

10. Show that the number of different possibilities for the outcomes of rolling n dice is $\binom{n+5}{n}$.

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