VI

*Linear programming and the theory of games

1. CONVEX SETS

An equation containing one or more variables will be called an *open statement*. For instance,

(a) $$-2x_1 + 3x_2 = 6$$

is an example of an open statement. If we let $A = (-2, 3)$, $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, and $b = 6$, we can write (a) in matrix form as

$$Ax = (-2, 3)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -2x_1 + 3x_2 = 6 = b.$$

For some two-component vectors x the statement $Ax = b$ is true and for others it is false. For instance, if $x = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ it is true since $-2 \cdot 3 + 3 \cdot 4 = 6$, and if $x = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ it is false since $-2 \cdot 2 + 3 \cdot 4 = 8$. The set of all two-component vectors x that make the open statement $Ax = b$ true is defined to be the *truth set* of the open statement.

Example 1. In plane geometry it is usual to picture in the plane the truth sets of open statements such as (a). Thus we can regard each two-component vector x as being the components of a point in the plane in the usual way. Then the truth set or *locus* (which is the geometric term for truth set) of (a) is the straight line plotted in Figure 1. Points on this line may be obtained by assuming values for one of the variables

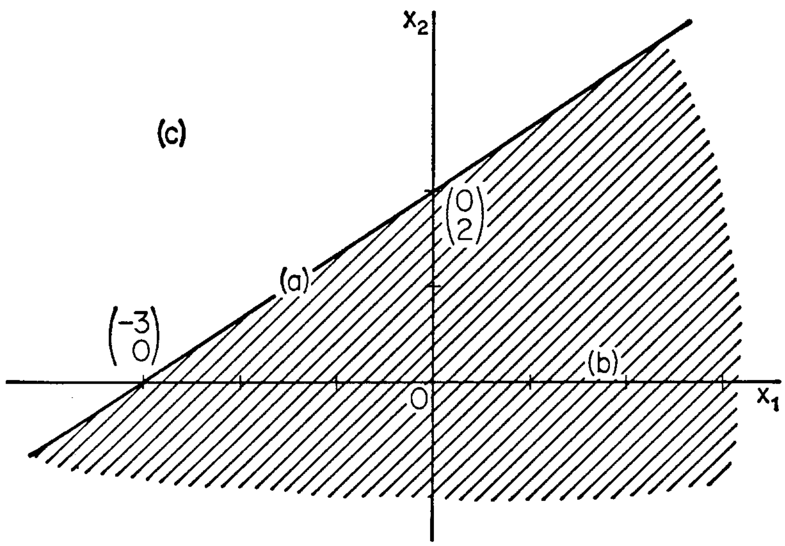


Figure 1

and computing the corresponding values for the other variable. Thus, setting $x_1 = 0$, we find $x_2 = 2$, so that the point $x = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$ lies on the locus; similarly, setting $x_2 = 0$, we find $x_1 = -3$, so that the point $\begin{pmatrix} -3 \\ 0 \end{pmatrix}$ lies on the locus, etc.

In the same way, inequalities of the form $Ax \leq b$ or $Ax < b$ or $Ax \geq b$ or $Ax > b$ are open statements and possess truth sets. And in the case that x is a two-component vector, these can be plotted in the plane.

Example 2. Consider the inequalities (b) $Ax < b$, (c) $Ax > b$, (d) $Ax \leq b$, and (e) $Ax \geq b$, where A, x, and b are as in Example 1. They may be written as

(b) $$-2x_1 + 3x_2 < 6$$
(c) $$-2x_1 + 3x_2 > 6$$
(d) $$-2x_1 + 3x_2 \leq 6$$
(e) $$-2x_1 + 3x_2 \geq 6.$$

Consider (b) first. What points $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ satisfy this inequality? By trial and error we can find many points on the locus. Thus the point $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is on it since $-2\cdot 1 + 3\cdot 2 = 4 < 6$; on the other hand, the point $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ is not on the locus because $-2\cdot 1 + 3\cdot 3 = -2 + 9 = 7$, which is not less than 6. In between these two points we find $\begin{pmatrix} 1 \\ \frac{8}{3} \end{pmatrix}$, which lies on the boundary, i.e., on the locus of (a). We note that, starting with $\begin{pmatrix} 1 \\ \frac{8}{3} \end{pmatrix}$ on locus (a), by increasing x_2 we went outside the locus (b); by decreasing x_2 we came into the locus (b) again. This holds in general. Given a point on the locus of (a), by increasing its second coordinate we get more than 6, but by decreasing the second coordinate we get less than 6, and hence the latter gives a point in the truth set of (b). Thus we find that the locus of (b) consists of all points of the plane *below* the line (a), in other words, the shaded area in Figure 1. The area on one side of a straight line is called an *open half plane.*

We can apply exactly the same analysis to show that the locus of (c) is the open half plane above the line (a). This can also be deduced from the fact that the truth sets of statements (a), (b), and (c) are disjoint and have as union the entire plane.

Since (d) is the disjunction of (a) and (b), the truth set of (d) is the union of the truth sets of (a) and (b). Such a set, which consists of an open half plane together with the points on the line that defines the half plane, is called a *closed half plane.* Obviously, the truth set of (e) consists of the union of (a) and (c) and therefore is also a closed half plane.

Frequently we want to assert several different open statements at once, that is, we want to assert the conjunction of several such statements. The easy way to do this is to let A be an $m \times n$ matrix, x an n-component column vector, and b an m-component column vector. Then the statement $Ax \leq b$ is the conjunction of the m statements $A_i x \leq b_i$ where A_i is the ith row of A and b_i is the ith entry of b.

Example 3. Consider the following example: $Ax \leq b$ where

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 2 & 3 \end{pmatrix}, \qquad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad b = \begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix}.$$

If we write the components of the equations $Ax \leq b$, we obtain

(f) $-x_1 \leq 0$ which is equivalent to $x_1 \geq 0$
(g) $-x_2 \leq 0$ which is equivalent to $x_2 \geq 0$
(h) $2x_1 + 3x_2 \leq 6.$

Here we are simultaneously asserting three different statements; i.e., we assert their conjunction. Therefore the truth set of $Ax \leq b$ is the intersection of the three individual truth sets. The truth set of (f) is the right half plane; the truth set of (g) is the upper half plane; and the truth set of (h) is the half plane below the line $2x_1 + 3x_2 = 6$. The intersection of these is the triangle (including the sides) shaded in Figure 2. The area shaded in Figure 2 contains those points which simultaneously satisfy (f), (g), and (h).

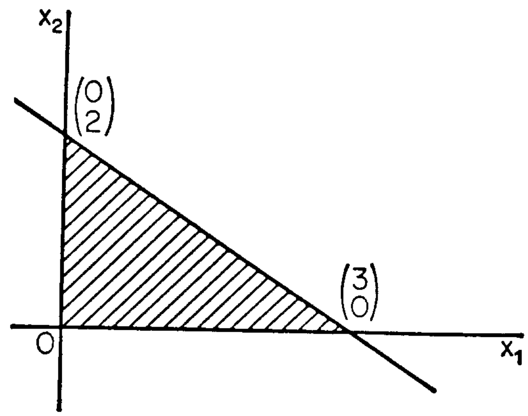


Figure 2

In the examples so far we have restricted ourselves to open statements with two variables. Such statements have truth sets that can be sketched in the plane. In the same way, open statements with three variables have truth sets that can be visualized in three-dimensional space. Open statements with four or more variables have truth sets in four or more dimensions, which we can no longer visualize. However, applied problems frequently lead to such statements. Fortunately, methods have been developed for handling them without having to visualize the truth sets geometrically. We shall illustrate these ideas in three-dimensional space, but everything that we do there can be extended without essential change to the general case of n variables.

In order to have a notation that will enable us to talk in general about conjunctions of m open statements in three dimensions, we shall consider x to be a three-component column vector, b an m-component column vector, and A an $m \times 3$ matrix. The ith row of A will be denoted by A_i for $i = 1, 2, \ldots, m$. Similarly, the ith component of b will be denoted by b_i. Of course, A_i is a three-component row vector and b_i is a number. We shall call the set of all three-component x vectors, *three-space*. Similarly, we call the set of all two-component x vectors, *two-space* or *the plane*.

We now set up some definitions for later use.

DEFINITION. The truth set of $A_i x = b_i$ is called a *plane* in three-space. The truth sets of inequalities of the form $A_i x < b_i$ or $A_i x > b_i$ are called *open half spaces*, while the truth sets of the inequalities $A_i x \leq b_i$ or $A_i x \geq b_i$ are called *closed half spaces* in three-space.

When we assert the conjunction of several open statements, the resulting truth set is the intersection of the truth sets of the individual open statements. Thus, in Example 3, we have the conjunction of $m = 3$ open statements in the plane. In Figure 2 we show this geometrically as the intersection of $m = 3$ closed half spaces (half planes) in two dimensions. Such intersections of closed half spaces are of special importance.

DEFINITION. The intersection of a finite number of closed half spaces is a *polyhedral convex set.*

The intuitive idea of polyhedral convex sets in two or three dimensions is very easy. In two dimensions they are sets, bounded by segments of straight lines that always "bulge out." For example, triangles, rectangles, pentagons, etc. are plane polyhedral (or polygonal) convex sets. In three dimensions they are sets, bounded by "pieces" of planes that always "bulge out." For instance, tetrahedra, cubes, octahedra, etc., are all such examples.

Theorem. Any polyhedral convex set is the truth set of an inequality statement of the form $Ax \leq b$.

Proof. A closed half space is the truth set of an inequality of the form $A_i x \leq b_i$. (An inequality of the form $A_i x \geq b_i$ can be converted into one of this form by multiplying by -1.) Now a polyhedral convex set is the truth set of the conjunction of several such statements. Since A is the matrix whose ith row is A_i and b is the column vector with components b_i, then the inequality statement $Ax \leq b$ is a succinct way of stating the conjunction of the inequalities $A_1 x \leq b_1, \ldots, A_m x \leq b_m$. This completes the proof.

The terminology polyhedral *convex* sets is used because these sets are special examples of convex sets. A convex set C is a set such that whenever u and v are points of C, the entire line segment between u and v also belongs to C. This is equivalent to saying that all points of the form $z = au + (1 - a)v$ for $0 \leq a \leq 1$ belong to C whenever u and v do. We shall be concerned primarily with polyhedral convex sets in this chapter.

EXERCISES

1. Draw pictures of the truth sets of $Ax \leq b$, where A and b are as given below. (Construct the truth sets of the individual statements first and then take their intersection.)

(a) $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & -3 \end{pmatrix}, \quad b = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}.$

(b) $A = \begin{pmatrix} -2 & -3 \\ -1 & 1 \\ 1 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} -6 \\ 2 \\ 3 \end{pmatrix}.$

(c) $A = \begin{pmatrix} 2 & 3 \\ -1 & 1 \\ 1 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 6 \\ 2 \\ 3 \end{pmatrix}.$

(d) $A = \begin{pmatrix} 0 & -1 \\ -1 & 0 \\ 1 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}.$

(e) $A = \begin{pmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 2 \\ 2 \\ 3 \\ 3 \end{pmatrix}.$

(f) $A = \begin{pmatrix} 3 & 2 \\ 3 & 2 \end{pmatrix}, \quad b = \begin{pmatrix} -6 \\ 6 \end{pmatrix}.$

(g) $A = \begin{pmatrix} -3 & -2 \\ 3 & 2 \end{pmatrix}, \quad b = \begin{pmatrix} -6 \\ 6 \end{pmatrix}.$

(h) $A = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$

(i) $A = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 2 \\ -5 \end{pmatrix}.$

(j) $A = \begin{pmatrix} -3 & -2 \\ -2 & -3 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} -6 \\ -6 \\ 0 \\ 0 \end{pmatrix}.$

(k) $A = \begin{pmatrix} -2 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} -7 \\ 0 \\ 0 \end{pmatrix}.$

2. Consider the following sets.

$\mathcal{U}$ is the whole plane;
A is the half plane which is the locus of $-2x_1 + x_2 < 3$.
B is the half plane which is the locus of $-2x_1 + x_2 > 3$.
C is the half plane which is the locus of $-2x_1 + x_2 \leq 3$.
D is the half plane which is the locus of $-2x_1 + x_2 \geq 3$.
L is the line which is the locus of $-2x_1 + x_2 = 3$.
$\mathcal{E}$ is the empty set.

Show that the following relationships hold among these sets: $\tilde{A} = D$, $\tilde{B} = C$, $\tilde{L} = A \cup B$, $C \cap D = L$, $A \cap B = \mathcal{E}$, $A \cap C = A$, $B \cap D = B$, $A \cup D = \mathcal{U}$, $B \cup C = \mathcal{U}$, $A \cup C = C$, $B \cup D = D$, $A \cup L = C$, $B \cup L = D$. Can you find other relationships?

3. Of the polyhedral convex sets constructed in Exercise 1, which have a finite area and which have infinite area?

[*Partial Ans.* (c), (d), (f), (h), and (j) are of infinite area; (g) is a line; (i) and (k) are empty.]

4. For each of the following half planes give an inequality of which it is the truth set.

(a) The open half plane above the x_1-axis. [*Ans.* $x_2 > 0$.]
(b) The closed half plane on and above the straight line making angles of 45° with the positive x_1- and x_2-axis.

Exercises 5–9 refer to a situation in which a retailer is trying to decide how many units of items A and B he should keep in stock. Let x be the number of units of A and y be the number of units of B. A costs \$4 per unit and B costs \$3 per unit.

5. One cannot stock a negative number of units of either A or B. Write these conditions as inequalities and draw their truth sets.

6. The maximum demand over the period for which the retailer is contemplating holding inventory will not exceed 600 units of A or 600 units of B. Modify the set found in Exercise 5 to take this into account.

7. The retailer is not willing to tie up more than \$2400 in inventory altogether. Modify the set found in Exercise 6.

8. The retailer decides to invest at least twice as much in inventory of item A as he does in inventory of item B. Modify the set of Exercise 7.

9. Finally, the retailer decides that he wants to invest $900 in inventory of item B. What possibilities are left? [*Ans.* None.]

10. Assume that the minimal nutritional requirements of human beings are given by the following table.

	Phosphorus	Calcium
Adult	.02	.01
Child	.03	.03
Infant	.01	.02

Plot the amount of phosphorus on the vertical axis and the amount of calcium on the horizontal. Then draw in the convex sets of minimal diet requirements for adults, children (noninfants), and infants. State whether or not the following assertions are true.

(a) If a child's needs are satisfied, so are an adult's.
(b) An infant's needs are satisfied only if a child's needs are.
(c) An adult's needs are satisfied only if an infant's needs are.
(d) Both an adult's and an infant's needs are satisfied only if a child's needs are.
(e) It is possible to satisfy adult needs without satisfying the needs of an infant.

11. Prove that the following sets are convex. Which are polyhedral convex sets?

(a) The interior plus the edges of a triangle.
(b) The interior of a circle.
(c) The interior of a rectangle.
(d) A rectangle surmounted by a semicircle.

12. Consider the plane with a cartesian coordinate system. A rectangle with sides of length a_1 and a_2 ($a_1 \neq a_2$) is placed with one corner at the origin and two of its sides along the axes. Prove that the interior of the rectangle plus its edges forms a polyhedral convex set and find the statement of the form $Ax \leq b$ of which it is the truth set.

13. The following polygons are placed in a plane with a cartesian coordinate system with one corner at the origin and one side along an axis. Find the statements $Ax \leq b$ of which they are the truth sets.

(a) A regular pentagon.
(b) A regular hexagon.

SUPPLEMENTARY EXERCISES

14. Consider the inequalities

(i)	$-x + 2y \leq 3$
(ii)	$x + y \leq 6$
(iii)	$x \geq 0$
(iv)	$y \geq 0$

as open statements, and vectors $\begin{pmatrix} x \\ y \end{pmatrix}$ as logical possibilities for these open statements.

(a) Sketch the truth set of each open statement, and also of their conjunction. Show that the statements are consistent by finding a logical possibility making all of them true.

(b) Show that the four statements cannot all be false.

15. How many regions would four independent statements yield? How many regions did we obtain in Exercise 14?

16. Add to the statements in Exercise 14, the statement

(v) $$3x + 4y \leq 22.$$

(a) Show that the statement, "If (i), (ii), (iii), and (iv) are true, then (v) is true," is logically true.

(b) Show that the convex set determined by the statements (i)–(v) is the same as that determined by (i)–(iv).

(c) Show that (a) and (b) are just two different ways of saying that (v) is unnecessary or superfluous in the determination of the convex set.

17. A manufacturer has two machines M_1 and M_2 which he uses to manufacture two products P_1 and P_2. To produce one unit of P_1, three hours of time on M_1 and six hours on M_2 are needed. And to produce one unit of P_2, six hours on M_1 and five hours on M_2 are needed. Assume that each machine can run a maximum of 2100 hours per year.

(a) Let x_1 be the number of units of P_1 and x_2 the number of units of P_2 produced. Write the inequality restrictions on $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

(b) Draw the convex set of possible production vectors x. (Save your work for later use.)

18. Two breakfast cereals, Krix and Kranch, supply varying amounts of vitamin B and iron; these are listed together with $\frac{1}{3}$ of the daily minimum requirements in the table below.

	Vitamin B	Iron
Krix	.15 mg./oz.	1.67 mg./oz.
Kranch	.10 mg./oz.	3.33 mg./oz.
$\frac{1}{3}$ Minimum requirements	.12 mg./day	2.0 mg./day

(a) Let w_1 be the amount of Krix eaten and w_2 the amount of Kranch eaten. Write the inequality restrictions on w_1 and w_2 in order that $\frac{1}{3}$ of the minimum daily requirements are met.

(b) Draw the convex set of possible amounts eaten defined by the inequalities of (a).

(c) What feasible diet requires a person to eat the fewest ounces of cereal?

[*Ans.* The diet requiring him to eat $\frac{6}{10}$ of an ounce of Krix and $\frac{3}{10}$ of an ounce of Kranch.]

19. Rework Exercise 18 under the assumption that a person wants to eat at least as much Kranch as Krix.

2. *MAXIMA AND MINIMA OF LINEAR FUNCTIONS*

In the present section we first discuss the problem of finding the extreme points of a bounded polyhedral convex set. Then we find out how to compute the maximum and minimum values of a linear function defined on such a set.

As in the preceding section, we use the following notation that is adapted for three-space, but which extends easily to any number of dimensions. The polyhedral convex set C is the truth set of the statement $Ax \leq b$ where A is an $m \times 3$ matrix, x is a three-component column vector, and b is an m-component column vector. We let $A_1, A_2, \ldots, A_m$ denote the rows of A, so that each A_i is a three-component row vector and

$$A = \begin{pmatrix} A_1 \\ A_2 \\ \cdot \\ \cdot \\ \cdot \\ A_m \end{pmatrix}.$$

The statement $Ax \leq b$ is then the conjunction of the statements

$$A_1x \leq b_1, A_2x \leq b_2, \ldots, A_mx \leq b_m.$$

DEFINITION. We shall call the truth set of the statement $A_i x = b_i$ the *bounding plane* of the half space $A_i x \leq b_i$. (In the two-dimensional case it is called the bounding *line*.)

Thus, in Figure 1 of the preceding section the slanting line (a) is the bounding line of the half space (b).

Sometimes it happens that one of the inequality statements defining a polyhedral convex set is unnecessary in the sense that the conjunction of the statements defining C is the same (equivalent) with or without the given statement. For instance, in Example 3 of Section 1, if we add the statement $x_1 \geq -1$ to the statements defining the convex set, it is superfluous, since the statement $x_1 \geq 0$ implies the statement $x_1 \geq -1$. But there are less obvious examples of superfluous statements, such as the one given in Exercise 16 of the preceding section. Still other examples are given in Exercise 1. Obviously, the elimination of superfluous inequalities does not change the polyhedral convex set C, and we assume that all such superfluous inequalities have been removed.

If the inequality $A_i x \leq b_i$ is not superfluous, then its bounding plane $A_i x = b_i$ must contain a point of the polyhedral convex set C. The *bounding planes* of C are the bounding planes of the (nonsuperfluous) half spaces of which C is the intersection.

In Example 3 of Section 1 the bounding planes (lines) of the convex set given there are the three boundary lines of the triangle shaded in Figure 2. Note that these lines intersect in pairs in three points, the vertices of the triangle. Such intersections are called *extreme points* of C. And in three dimensions, if T is a point of C that is the intersection of three bounding planes of C, then it is an *extreme point* of C.

Example 1. Find the extreme points of the polyhedral convex set $Ax \leq b$ where

$$A = \begin{pmatrix} -2 & -1 \\ 1 & -3 \\ 1 & 2 \end{pmatrix}, \qquad b = \begin{pmatrix} 9 \\ 6 \\ 3 \end{pmatrix}.$$

A sketch of the three half planes, Figure 3, shows that the set is a triangle. Hence we can find the extreme points by changing the inequalities to equalities in pairs and solving three sets of simultaneous equations. We obtain in this way the points

$$\begin{pmatrix} -3 \\ -3 \end{pmatrix}, \quad \begin{pmatrix} -7 \\ 5 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} \frac{21}{5} \\ -\frac{3}{5} \end{pmatrix},$$

which are the extreme points of the set.

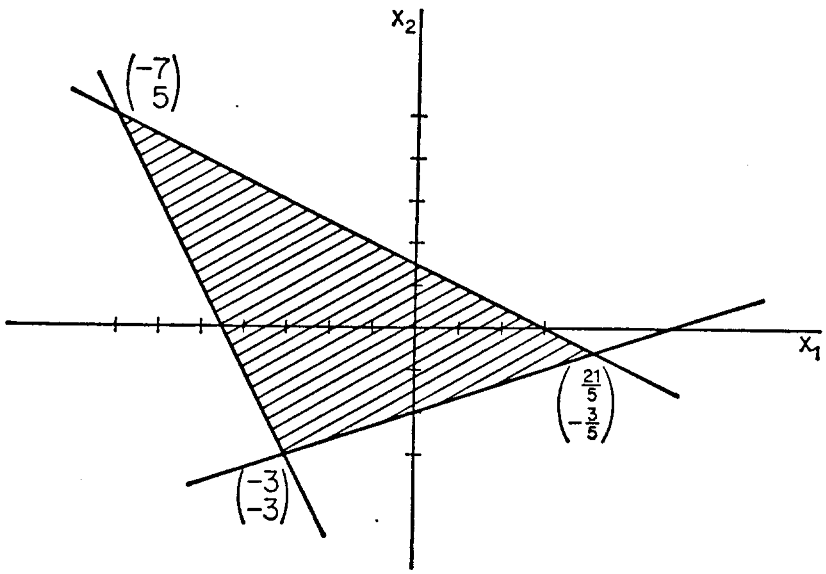


Figure 3

We can now give an interpretation for the various points of the polyhedral convex set in terms of the system of inequalities. An extreme point, in the plane, lies on two boundaries, which means that two of the inequalities are actually equalities. A point on a side, other than an extreme point, lies on one boundary and hence one inequality is an equality. An interior point of the polygon must, by a process of elimination, correspond to the case where the inequalities are all strict inequalities, i.e., not only $\leq$ but $<$ holds.

There is a mechanical (but lengthy) method for finding all the extreme points of a polyhedral convex set C in three-space defined by $Ax \leq b$. Consider the bounding hyperplanes $A_1x = b_1, \ldots, A_mx = b_m$ of the half spaces that determine C. Select a subset of three of these hyperplanes and solve their equations simultaneously. If the result is a unique point x^0 (and only then), check to see whether or not x^0 belongs to C. If it does, by the above definition, x^0 is an extreme point of C. Moreover, all extreme points of C can be found in this manner.

Example 2. Let

$$A = \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix}0\\0\end{pmatrix}.$$

Then the polyhedral convex set C defined by $Ax \leq b$ is the first quadrant of the x_1, x_2 plane, shaded in Figure 4. The only extreme point is the origin, which is the intersection of the lines $x_1 = 0$ and $x_2 = 0$. This is an example of an *unbounded* polyhedral convex set.

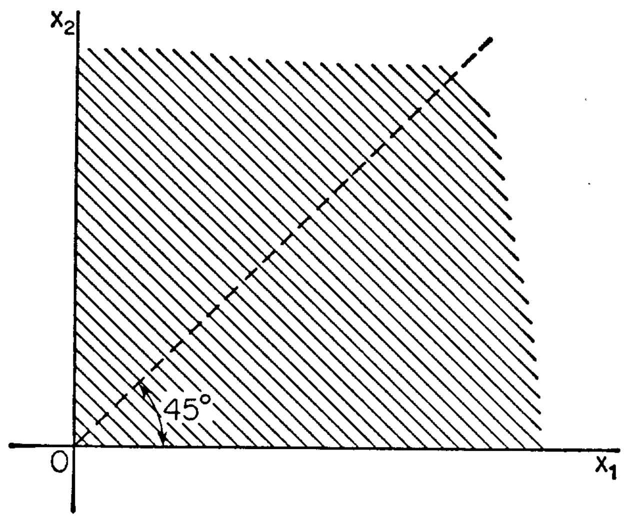


Figure 4

Notice that the set C contains the *ray* or half line that starts at the origin of coordinates and extends upward to the right making a 45° angle with the axes. This ray is dotted in Figure 4. Of course, this set also contains many other rays.

We shall say that a polyhedral convex set is *bounded* if it does not contain a ray. A set, such as the one in Figure 4, that does contain rays will be called *unbounded.* For simplicity we shall restrict our discussion to bounded convex sets in most of this chapter. In particular, this means that necessarily $m > n$, that is, the convex set must be the intersection of at least $n + 1$ half spaces. Thus we need at least three lines in the plane, and at least four planes in three-space to produce a bounded set. This is a necessary but not sufficient condition that the convex set is bounded (see Exercise 23).

Example 3. Let us suppose that in a business problem x_1 and x_2 are quantities we can control, except that there are limitations imposed which can be stated as inequalities. We shall assume that the system of inequalities given in Example 1 limits our choice of x_1 and x_2. Let us assume that a given choice of x_1 and x_2 results in a profit of $x_1 + 2x_2$ dollars. What is the most and the least profit we can make? We must find the maximum and the minimum value of $x_1 + 2x_2$ for points (x_1, x_2) in the triangle. Let us first try the extreme points. At $(-3, -3)$ we would have a profit of -9, i.e., a loss of \$9. At $(-7, 5)$ we have a profit of \$3, and at $(\frac{21}{5}, -\frac{3}{5})$ also a profit of \$3. What can we say about the remainder of the triangle? The last inequality tells that $x_1 + 2x_2 \leq 3$, hence our profit cannot be more than \$3. If we multiply the first inequality by $\frac{5}{7}$ and the second by $\frac{3}{7}$ and add them, we find that $x_1 + 2x_2 \geq -9$; hence, we cannot lose more than \$9. We have thus shown that both the greatest profit and the greatest loss occur at an extreme point. We will show that this is true in general.

Given a polyhedral convex set C and a linear function

$$cx = c_1x_1 + c_2x_2 + \ldots + c_nx_n,$$

where $c = (c_1, c_2, \ldots, c_n)$, we want to show in general that the maximum and minimum values of the function cx always occur at extreme points of C. We shall carry out the proof for the planar case in which $n = 2$, but our results are true in general.

First, we will show that the values of the linear function $c_1x_1 + c_2x_2$ on any line segment lie *between* the values the function has at the two end points (possibly equal to the value at one end point). We represent the points as column vectors $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and then we see that our linear function is represented by the row vector $c = (c_1, c_2)$. Let the end points of the segment be

$$p = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} \quad \text{and} \quad q = \begin{pmatrix} x_1'' \\ x_2'' \end{pmatrix}.$$

We have seen in Chapter V (see Figure 4) that the points in between p and q can be represented as $tp + (1 - t)q$, with $0 \leq t \leq 1$. If the values of the function at the points p and q are P and Q, respectively (assume that $P \geq Q$), then at a point in between the value will be $tP + (1 - t)Q$, since the function is linear. This value can also be written as

$$tP + (1 - t)Q = Q + (P - Q)t = P - (1 - t)(P - Q),$$

which (for $0 \leq t \leq 1$) is at least Q and at most P.

We are now in a position to prove the result illustrated in Example 3.

Theorem. A linear function cx defined over a polyhedral convex set C takes on its maximum (and minimum) value at an extreme point of C.

The proof of the theorem is illustrated in Figure 5. We shall suppose that at the extreme point p the function takes on a value P greater than or equal to the value at any other extreme point, and at the extreme point q it takes on its smallest extreme point value, Q. Let r be any

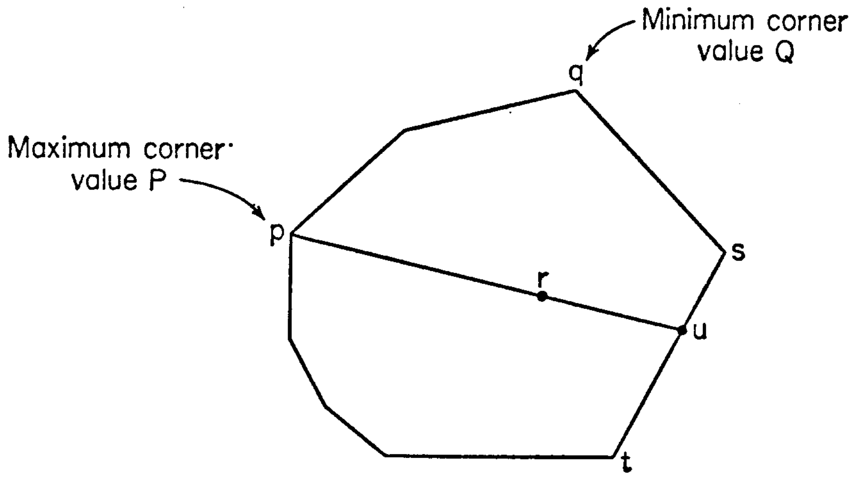


Figure 5

point of the polygon. Draw a straight line between p and r and continue it until it cuts the polygon again at a point u lying on an edge of the polygon, say the edge between the corner points s and t. (The line may even cut the edge at one of the points s and t; the analysis remains unchanged.) By hypothesis the value of the function at any corner point must lie between Q and P. By the above result the value of the function at u must lie between its values at s and t, and hence must also lie between Q and P. Again by the above result the value of the function at r must lie between its values at p and u, and hence must also lie between Q and P. Since r was any point of the polygon, our theorem is proved.

Suppose that in place of the linear function $c_1x_1 + c_2x_2$ we had considered the function $c_1x_1 + c_2x_2 + k$. The addition of the constant k merely changes every value of the function, including the maximum and minimum values of the function, by that amount. Hence the analysis of where the maximum and minimum values of the function are taken on is unchanged. Therefore, we have the following theorem.

Theorem. The function $cx + k$ defined over a polyhedral convex set C takes on its maximum (and minimum) value at an extreme point of C.

A method of finding the maximum or minimum of the function $cx + k$ defined over a convex set C is then the following: Find the extreme points of the set; there will be a finite number of them. Substitute the coordinates of each into the function. The largest of the values so obtained will be the maximum of the function and the smallest value will be the minimum of the function. The method is illustrated in Example 3 above.

EXERCISES

1. In the following sets of inequalities at least one is superfluous. In each case find the superfluous ones.

(a)
$$\begin{aligned} x_1 + x_2 &\le 3 \\ -x_1 - x_2 &\ge 0 \\ x_1 &\ge -1 \\ -x_2 &\le 2. \end{aligned}$$

(b)
$$\begin{aligned} x_1 + x_2 &\ge 0 \\ x_1 - x_2 &\le 0 \\ x_1 &\le 4 \\ x_2 &\ge -4. \end{aligned}$$

(c)
$$\begin{aligned} -1 \le x_1 &\le 1 \\ -2 \le x_2 &\le 2 \\ x_1 + x_2 &\ge -10 \\ 2x_1 - x_2 &\le 2. \end{aligned}$$

[*Ans.* (a) $x_1 + x_2 \le 3$.]

2. (a) Draw a picture of the convex set defined by the inequalities

$$\begin{aligned} 2x_1 + x_2 + 9 &\le 0 \\ -x_1 + 3x_2 + 6 &\le 0 \\ x_1 + 2x_2 - 3 &\le 0. \end{aligned}$$

(b) What is the relationship between this and Figure 3?

3. Find the corner points of the convex polygons given in parts (a), (b), and (e) of Exercise 1 of Section 1.

[*Ans.* (a) $\begin{pmatrix} 3 \\ -2 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \begin{pmatrix} -3 \\ 2 \end{pmatrix}$; (e) $\begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} -2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \end{pmatrix}, \begin{pmatrix} -2 \\ -3 \end{pmatrix}$.]

4. (a) Show that the three lines whose equations are

$$\begin{aligned} 2x_1 + x_2 + 9 &= 0 \\ -x_1 + 3x_2 + 6 &= 0 \\ x_1 + 2x_2 - 3 &= 0 \end{aligned}$$

divide the plane into seven convex regions. Mark these regions with Roman numerals I–VII.

(b) For each of the seven regions found in part (a), write a set of three inequalities, having the region as its locus. [*Hint:* Two of these sets of inequalities are considered in Exercise 2.]

(c) There is one more way of putting inequality signs into the three equations given in (a). What is the locus of this last set of inequalities? [*Ans.* The empty set ε.]

5. A convex polygon has the points $(-1, 0)$, $(3, 4)$, $(0, -3)$, and $(1, 6)$ as extreme points. Find a set of inequalities which defines the convex polygon having these extreme points.

6. Find the extreme points of the convex polygon given by the equations

$$\begin{aligned} 2x_1 + x_2 + 9 &\geq 0 \\ -x_1 + 3x_2 + 6 &\geq 0 \\ x_1 + 2x_2 - 3 &\leq 0 \\ x_1 + x_2 &\leq 0. \end{aligned}$$

[*Hint:* Use some of the results of Example 1 in the text.]

7. Find the extreme values of the function G defined by

$$G(x) = 7x_1 + 5x_2 - 3$$

over the convex polygon of Exercise 6.

8. Find the maximum and minimum of the function

$$G(x) = -2x_1 + 5x_2 + 17$$

over each of the convex polygons given in parts (a), (b), and (e) of Exercise 1 of Section 1. [*Ans.* (a) 33, 1; (e) 36, −2.]

9. Find the maximum and minimum, when they exist, of the function

$$G(x) = 5x_1 + 3x_2 - 6$$

over each of the polyhedral convex sets given in parts (h) and (j) of Exercise 1 of Section 1. [*Ans.* (h) Neither maximum nor minimum; (j) minimum is 3.]

10. The owner of an oil truck with a capacity of 500 gallons hauls gasoline and oil products from city to city. On any given trip he wishes to load his truck with at least 200 gallons of regular test gasoline, at least 100 gallons of high test gasoline, and at most 150 gallons of kerosene. Assuming that he always fills his truck to capacity, find the convex set of ways that he can load his truck. Interpret the extreme points of the set. [*Hint:* There are four extreme points.]

11. An advertiser wishes to sponsor a half hour television comedy and must decide on the composition of the show. The advertiser insists that there be at least three minutes of commercials, while the television network requires that the commercial time be limited to at most 15 minutes. The comedian refuses to work more than 22 minutes each half hour show. If a band is added to the show to play during the time that neither the comedian nor the commercials are on, construct the convex set C of possible assignments of time to the comedian, the commercials, and the band that use up the 30 minutes. Find the extreme points of C.

[*Ans.* If x_1 is the comedian time, x_2 the commercial time, and $30 - x_1 - x_2$ the band time, the extreme points are

$$\begin{pmatrix}0\\3\end{pmatrix}, \begin{pmatrix}22\\3\end{pmatrix}, \begin{pmatrix}22\\8\end{pmatrix}, \begin{pmatrix}15\\15\end{pmatrix} \text{ and } \begin{pmatrix}0\\15\end{pmatrix}.]$$

12. In Exercise 10 suppose that the oil truck operator gets 3 cents per gallon for delivering regular gasoline, 2 cents per gallon for high test, and 1 cent per gallon for kerosene. Write the expression that gives the total amount he will be paid for each possible load that he carries. How should he load his truck in order to earn the maximum amount?

[*Ans.* He should carry 400 gallons of regular gasoline, 100 gallons of high test, and no kerosene.]

13. In Exercise 12, if he gets 3 cents per gallon of regular and 2 cents per gallon of high test gasoline, how high must his payment for kerosene become before he will load it on his truck in order to make a maximum profit?

[*Ans.* He must get paid at least 3 cents per gallon of kerosene.]

14. In Exercise 11 let x_1 be the number of minutes the comedian is on and x_2 be the number of minutes the commercial is on the program. Suppose the comedian costs $200 per minute, the commercials cost $50 per minute, and the band is free. How should the advertiser choose the composition of the show in order that its cost be a minimum?

15. Consider the polyhedral convex set P defined by the inequalities

$$-1 \leq x_1 \leq 4$$
$$0 \leq x_2 \leq 6.$$

Find four different sets of conditions on the constants a and b that the function $F(x) = ax_1 + bx_2$ should have its maximum at one and only one of the four corner points of P. Find conditions that F should have its minimum at each of these points.

[*Ans.* For example, the maximum is at $\begin{pmatrix}4\\6\end{pmatrix}$ if $a > 0$ and $b > 0$.]

16. Let H be the quadratic function defined by $H(x) = (x_1 - \frac{1}{4})^2 + (x_2 - \frac{1}{4})^2$ on the convex set C which is the truth set of the inequalities

$$x_1 + x_2 \leq 1, \qquad x_1 \geq 0, \qquad x_2 \geq 0.$$

Are the maximum and minimum values of H taken on at the extreme points of C? Discuss reasons why this problem is essentially harder than that of finding the extreme values of a linear function on a polyhedral convex set.

17. A set of points is said to be convex if whenever it contains two points it also contains the line segment connecting them. Show that

(a) If two points are in the truth set of an inequality, then any point on the connecting segment is also in the truth set.
(b) Every polygonal convex set is a convex set in the above-mentioned sense.

18. Give an example of a quadrilateral that is not a convex set.

19. Prove that for any three vectors, u, v, w, the set of all points $au + bv + cw$ ($a \geq 0$, $b \geq 0$, $c \geq 0$, $a + b + c = 1$) is a convex set. What geometric figure is this locus? [*Ans.* In general, the locus is a triangle.]

20. Let C be any plane polyhedral convex set. Show that if x is a point that lies on three bounding lines of C, then one of the inequalities defining C is superfluous.

21. Let x and y be two distinct points of a polyhedral convex set C, let t be a number such that $0 < t < 1$, and define $z = tx + (1 - t)y$. Show that z is not an extreme point of C.

22. Prove that the intersection of two half planes is a bounded convex set only if it is empty.

23. Construct examples that show that the intersection of three half planes either may or may not be a bounded convex set.

SUPPLEMENTARY EXERCISES

24. In Exercise 17 of Section 1 assume that the manufacturer makes a profit of \$4 for each unit of P_1 and \$5 for each unit of P_2. How many units of each should he produce in order to maximize his profit? What is his maximum profit?

[*Ans.* 100 units of P_1 and 300 units of P_2; his maximum profit is \$1900.]

25. In Exercise 18 of Section 1 assume that Krix costs $\frac{5}{3}$ of a cent per ounce and Kranch costs $\frac{5}{2}$ of a cent per ounce. In order to satisfy $\frac{1}{3}$ of the daily minimum requirements at minimum cost, how many ounces of each cereal should a person eat? What is the cost of the minimum cost diet?

[*Partial Ans.* The cost is $\frac{7}{4}$ of a cent per day.]

26. Rework Exercise 25 under the assumption that a person wants to eat at least as much Kranch as Krix (see Exercise 19 of Section 1).

27. An automobile manufacturer has 900 tons of metal on hand from which he is to make x_1 automobiles and x_2 trucks. It takes 2 tons of metal and 200 man-hours of work to make an automobile, and it takes 4 tons of metal and 150 man-hours of work to make a truck. He has 60,000 man-hours of time available. If he makes a profit of \$500 on an automobile and \$800 on a truck, how many of each should he make to maximize his profit?

(a) Set up the inequality constraints on the variables.
(b) Draw the convex set of feasible vectors.
(c) Find the optimal production vector and the maximum profit.
[*Ans.* He should produce 210 automobiles and 120 trucks for a maximum profit of $201,000.]

28. Suppose in Exercise 27 that the profit on automobiles drops to $350. How will this affect the production and profits?

[*Ans.* He produces only trucks for profit of $180,000.]

29. Suppose in Exercise 27 that the profit on trucks drops to $350. How should the manufacturer now produce?

3. *LINEAR PROGRAMMING PROBLEMS*

An important class of practical problems are those which require the determination of the maximum or the minimum of a linear function $cx + k$ defined over a polyhedral convex set of points C. We illustrate these so-called *linear programming problems* by means of the following examples.*

Example 1. An automobile manufacturer makes automobiles and trucks in a factory that is divided into two shops. Shop 1, which performs the basic assembly operation, must work five man-days on each truck but only two man-days on each automobile. Shop 2, which performs finishing operations, must work three man-days for each automobile or truck that it produces. Because of men and machine limitations, shop 1 has 180 man-days per week available while shop 2 has 135 man-days per week. If the manufacturer makes a profit of $300 on each truck and $200 on each automobile, how many of each should he produce to maximize his profit?

To state the problem mathematically, we set up the following notation: Let x_1 be the number of trucks and x_2 the number of automobiles to be produced per week. Then these quantities must satisfy the following restrictions.

$$5x_1 + 2x_2 \leq 180$$

$$3x_1 + 3x_2 \leq 135.$$

* Readers interested in an elementary treatment of the simplex method of linear programming are referred to Kemeny, Schleifer, Snell, and Thompson, *Finite Mathematics with Business Applications* (Englewood Cliffs, N.J.: Prentice-Hall, Inc., 1962), pp. 384–401.

We want to maximize the linear function $300x_1 + 200x_2$, subject to these inequality constraints, together with the obviously necessary constraints that $x_1 \geq 0$ and $x_2 \geq 0$.

To further simplify notation, we define the quantities

$$A = \begin{pmatrix} 5 & 2 \\ 3 & 3 \end{pmatrix}, \quad b = \begin{pmatrix} 180 \\ 135 \end{pmatrix} \quad \text{and} \quad c = (300, 200).$$

Then we can state this linear programming problem as follows.

Maximum problem: Determine the vector x so that the weekly profit, given by the quantity cx, is a maximum, subject to the inequality constraints $Ax \leq b$ and $x \geq 0$. The inequality constraints insure that the weekly number of available man-hours is not exceeded and that non-negative quantities of automobiles and trucks are produced.

The graph of the convex set of possible x vectors is pictured in Figure 6. Clearly this is a problem of the kind discussed in the previous section.

The extreme points of the convex set C are

$$T_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 36 \\ 0 \end{pmatrix}, \quad T_3 = \begin{pmatrix} 0 \\ 45 \end{pmatrix} \quad \text{and} \quad T_4 = \begin{pmatrix} 30 \\ 15 \end{pmatrix}.$$

Following the solution procedure outlined in the previous section, we

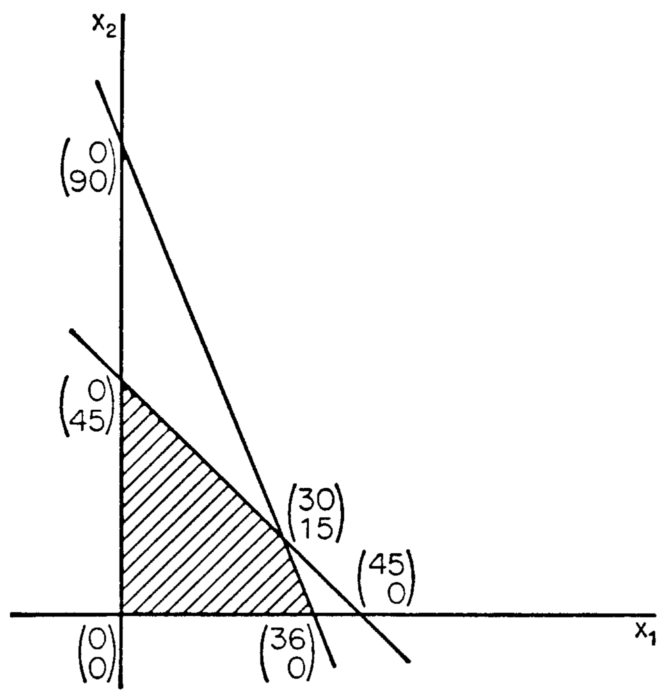


Figure 6

test the function $cx = 300x_1 + 200x_2$ at each of these extreme points. The values taken on are 0, 10,800, 9000, and 12,000. Thus the maximum weekly profit is \$12,000 and is achieved by producing 30 trucks and 15 automobiles per week.

Example 2. A mining company owns two different mines that produce a given kind of ore. The mines are located in different parts of the country and hence have different production capacities. After crushing, the ore is graded into three classes: high-grade, medium-grade, and low-grade ores. There is some demand for each grade of ore. The mining company has contracted to provide a smelting plant with 12 tons of high-grade, eight tons of medium-grade, and 24 tons of low-grade ore per week. It costs the company \$200 per day to run the first mine and \$160 per day to run the second. However, in a day's operation the first mine produces six tons of high-grade, two tons of medium-grade, and four tons of low-grade ore, while the second mine produces daily two tons of high-grade, two tons of medium-grade, and 12 tons of low-grade ore. How many days a week should each mine be operated in order to fulfill the company's orders most economically?

Before solving the problem it is convenient to summarize the above information as in the tableau of Figure 7. The numbers in the tableau form a 2-by-3 matrix, the requirements form a row vector c, and the

	High-grade ore	Medium-grade ore	Low-grade ore	b
Mine 1	6	2	4	\$200
Mine 2	2	2	12	\$160
c	12	8	24	

Figure 7

costs form a column vector b. The entries in the matrix indicate the production of each kind of ore by the mines, the entries in the requirements vector c indicate the quantities that must be produced, and the entries in the cost vector b indicate the daily costs of running each mine.

Let $w = (w_1, w_2)$ be the two-component row vector whose component w_1 gives the number of days per week that mine 1 operates and w_2 gives the number of days per week that mine 2 operates. If we define the quantities

$$A = \begin{pmatrix} 6 & 2 & 4 \\ 2 & 2 & 12 \end{pmatrix}, \qquad c = (12, 8, 24), \quad \text{and} \quad b = \begin{pmatrix} 200 \\ 160 \end{pmatrix},$$

we can state the above problem as a minimum problem.

Minimum problem: Determine the vector w so that the weekly operating cost, given by the quantity wb, is a minimum, subject to the inequality restraints $wA \geq c$ and $w \geq 0$. The inequality restraints insure that the weekly output requirements are met and the limits on the components of w are not exceeded.

It is clear that this is a minimum problem of the type discussed in detail in the preceding section. In Figure 8 we have graphed the convex

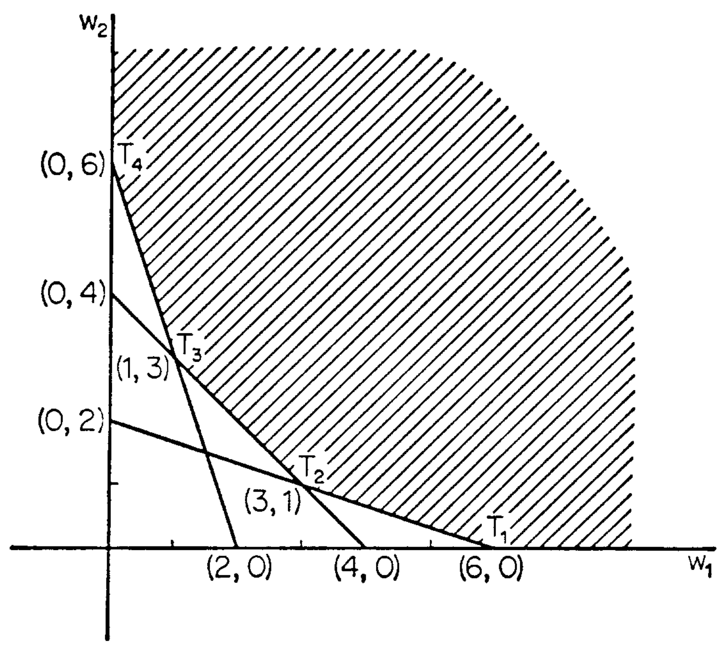


Figure 8

polyhedral set C defined by the inequalities $wA \geq c$. (We have omitted the additional obvious constraints $w_1 \leq 7$ and $w_2 \leq 7$, for simplicity. These, if added, would make the convex set bounded.)

The extreme points of the convex set C are

$$T_1 = (6, 0), \quad T_2 = (3, 1), \quad T_3 = (1, 3), \quad T_4 = (0, 6).$$

Testing the function $wb = 200w_1 + 160w_2$ at each of these extreme points, we see that it takes on the values 1200, 760, 680, and 960, respectively. We see that the minimum operating cost is \$680 per week and it is achieved at T_3, i.e., by operating the first mine one day per week and the second mine three days a week.

Observe that if the mines are operated as indicated, then the combined weekly production will be 12 tons of high-grade ore, 8 tons of medium-grade ore, and 40 tons of low-grade ore. In other words, for this solution, low-grade ore is overproduced. If the company has no other demand for the low-grade ore, then it must discard 16 tons of it per week in this minimum-cost solution of its production problem.

Example 3. As a variant of Example 2, assume that the cost vector is $b = \begin{pmatrix}160\\200\end{pmatrix}$; in other words, the first mine now has a lower daily cost than the second. By the same procedure as above we find that the minimum cost level is again \$680 and is achieved by operating the first mine three days a week and the second mine one day per week. In this solution, 20 tons of high-grade ore, instead of the required 12 tons, are produced, while the requirements of medium- and low-grade ores are exactly met. Thus eight tons of high-grade ore must be discarded per week.

Example 4. As another variant of Example 2, assume that the cost vector is $b = \begin{pmatrix}200\\200\end{pmatrix}$; in other words, both mines have the same production costs. Evaluating the cost function wb at the extreme points of the convex set, we find costs of \$1200 on two of the extreme points (T_1 and T_4) and costs of \$800 on the other two extreme points (T_2 and T_3). Thus the minimum cost is attained by operating either one of the mines three days a week and the other one one day a week. But there are other solutions, since if the minimum is taken on at two distinct extreme points, it is also taken on at each of the points on the line segment between. Thus any vector w where $1 \leq w_1 \leq 3$, $1 \leq w_2 \leq 3$, and $w_1 + w_2 = 4$ also gives a minimum-cost solution. For example, each mine could operate two days a week.

It can be shown (see Exercise 2) that for any solution w with $1 < w_1 < 3$, $1 < w_2 < 3$, and $w_1 + w_2 = 4$, both high-grade and low-grade ores are overproduced.

EXERCISES

1. In Example 1, assume that profits are \$200 per truck and \$300 per automobile. What should the factory now produce for maximum profit?

2. In Example 4, show that both high- and low-grade ores are overproduced for solution vectors w with $1 < w_1 < 3$, $1 < w_2 < 3$, and $w_1 + w_2 = 4$.

3. A well-known nursery rhyme says "Jack Sprat could eat no fat. His wife (call her Jill) could eat no lean . . . ." Suppose Jack wishes to have at least one pound of lean meat per day, while Jill needs at least .4 pound of fat per day. Assume they buy only beef having 10 per cent fat and 90 per cent lean, and pork having 40 per cent fat and 60 per cent lean. Jack and Jill want to fulfill their minimal diet requirements at the lowest possible cost.

(a) Let x be the amount of beef and y the amount of pork which they purchase per day. Construct the convex set of points in the plane representing purchases that fulfill both persons' minimum diet requirements.

(b) Suggest necessary restrictions on the purchases that will change this set into a convex polygon.

(c) If beef costs \$1 per pound, and pork costs 50 cents per pound, show that the diet of least cost has only pork, and find the minimum cost. [*Ans.* 83 cents.]

(d) If beef costs 75 cents and pork costs 50 cents per pound, show that there is a whole line segment of solution points and find the minimum cost. [*Ans.* 83 cents.]

(e) If beef and pork each cost \$1 a pound, show that the unique minimal cost diet has both beef and pork. Find the minimum cost. [*Ans.* \$1.40.]

(f) Show that the restriction made in part (b) did not alter the answer given in (c)–(e).

4. In Exercise 3(d) show that for all but one of the minimal cost diets Jill has more than her minimum requirement of fat, while Jack always gets exactly his minimal requirement of lean. Show that all but one of the minimal cost diets contain some beef.

5. In Exercise 3(e) show that Jack and Jill each get exactly their minimal requirements.

6. In Exercise 3, if the price of pork is fixed at \$1 a pound, how low must the price of beef fall before Jack and Jill will eat only beef? [*Ans.* 25 cents.]

7. In Exercise 3, suppose that Jack decides to reduce his minimal requirement to .6 pound of lean meat per day. How does the convex set change? How do the solutions in 3(c), (d), and (e) change?

8. A poultry farmer raises chickens, ducks, and turkeys and has room for 500 birds on his farm. While he is willing to have a total of 500 birds, he does not want more than 300 ducks on his farm at any one time. Suppose that a chicken costs \$1.50, a duck \$1.00, and a turkey \$4.00 to raise to maturity. Assume that the farmer can sell chickens for \$3.00, ducks for \$2.00, and turkeys for T dollars each. He wants to decide which kind of poultry to raise in order to maximize his profit.

(a) Let x be the number of chickens and y be the number of ducks he will raise. Then $500 - x - y$ is the number of turkeys he raises. What is the convex set of possible values of x and y which satisfy the above restrictions?

(b) Find the expression for the cost of raising x chickens, y ducks, and $(500 - x - y)$ turkeys. Find the expression for the total amount he gets for these birds. Compute the profit which he would make under these circumstances.

(c) If $T = \$6.00$, show that to obtain maximal profit the farmer should raise only turkeys. What is the maximum profit? [*Ans.* \$1000.]

(d) If $T = \$5.00$, show that he should raise only chickens and find his maximum profit. [*Ans.* \$750.]

(e) If $T = \$5.50$, show that he can raise any combination of chickens and turkeys and find his maximum profit. [*Ans.* \$750.]

9. Rework Exercise 8 if the price of chickens drops to \$2.00 and T is (a) \$6.00, (b) \$5.00, (c) \$4.50, and (d) \$4.00.

10. In Exercise 8 show that if the price of turkeys drops below \$5.50, the farmer should raise only chickens. Also show that if the price is above \$5.50, he should raise only turkeys.

11. In Exercise 10 of Section 2, assume that the truck operator gets p cents a gallon for regular gasoline, q cents a gallon for high-test gasoline, and r cents a gallon for kerosene. Show that he will carry kerosene for maximum profit only if $r \geq p$ and $r \geq q$.

12. In Exercise 11 of Section 2, suppose that for each minute the comedian is on the program 70,000 more people will tune in, for each minute the band is on 10,000 more people will tune in, and for each minute the commercial is on one more person will tune in. Let N be the function that gives the number of persons that tune in for each point in C. How should the times be allotted in order that N be a maximum?

[*Ans.* There should be 3 minutes of commercials, 22 minutes of the comedian, and 5 minutes of band music.]

13. In Exercise 11 of Section 2, assume that the band and comedian each cost \$200 per minute while the commercials cost \$50 per minute. Write the function that gives the cost of the program. Show that there is a whole line segment of minimum cost solutions.

[*Ans.* The commercials are on for 15 minutes while the band and comedian can share the remaining 15 minutes in any manner.]

SUPPLEMENTARY EXERCISES

14. Suppose that 1 unit of hog liver contains 1 unit of carbohydrates, 3 units of vitamins, and 3 units of proteins and costs 50 cents per unit. Suppose 1 unit of castor oil contains 3, 4, and 1 units of these items, respectively, and costs 25 cents per unit. If hog liver and castor oil are the only foods available, and the minimum daily requirements are 8 units of carbohydrates, 19 units of vitamins, and 7 units of proteins, find the minimum cost diet, using the following procedure.

(a) Let w_1 be the number of units of hog liver and w_2 the number of units of castor oil purchased. Set up the inequality constraints that these variables must satisfy.

(b) Find the objective function to be minimized.

(c) Sketch the convex set of possible food purchases.

(d) Locate the extreme point giving the minimum cost diet.

[*Ans.* Buy one unit of hog liver and four units of castor oil; cost, \$1.50.]

15. Suppose that the minimum cost diet found in Exercise 14 is found to be unpalatable. In order to increase its palatability, add a constraint requiring that at least three units of hog liver be purchased, and re-solve the problem. How much is the cost of the minimum cost diet increased due to this palatability requirement? [*Ans.* \$.63.]

16. A farmer owns a 100 acre farm and can plant any combination of two crops I and II. Crop I requires one man-day of labor and \$10 of capital for each acre planted, while crop II requires 4 man-days of labor and \$20 of capital for each acre planted. Crop I produces \$40 of net revenue per acre and crop II produces \$60 net revenue per acre. The farmer has \$1100 of capital and 160 man-days of labor available for the year.

(a) Let x_1 be the number of acres of crop I and x_2 the number of acres of crop II planted. Set up the inequality constraints.

(b) Set up the expression that gives the net revenue from a planting scheme $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

(c) Sketch the convex set of possible planting schemes.

(d) Find the extreme point that gives the maximum revenue.

[*Partial Ans.* The maximum revenue is \$4200.]

17. In Exercise 16 assume that the revenue from crop II is \$90 per acre.
   (a) Find the new maximum revenue scheme, and show that now the best thing for the farmer to do is to leave 15 acres unplanted.
   (b) Explain why the farmer should leave part of his land fallow in this case.

18. A manufacturer produces two types of bearings, A and B, utilizing three types of machines, lathes, grinders, and drill presses. The machinery requirements for one unit of each product, in hours, is expressed in the following table.

Bearing	*Lathe*	*Grinder*	*Drill Press*
A	.01	.03	.03
B	.02	.01	.015
Weekly machine capacity (hours)	400	450	480

The weekly machine capacities are also shown in the table. He makes a profit of 10 cents per type A bearing and 15 cents per type B bearing.
   (a) Let x_1 be the number of type A bearings and x_2 the number of type B bearings produced. Write the inequality restrictions on these variables.
   (b) Write the objective function to be maximized.
   (c) Draw the convex set of possible production plans.
   (d) Find the optimum production plan.
   [*Ans.* He should produce 8000 type A and 16,000 type B bearings for a profit of \$3200.]

19. In Exercise 18 assume that the manufacturer has enough money to purchase one more machine of any kind, and that doing so will increase the capacity of that type of machine by 40 hours per week.
   (a) He wants to buy a new grinder. Would you advise him to do so? [*Ans.* No.]
   (b) Which machine would you recommend that he buy? [*Ans.* A lathe.]

4. STRICTLY DETERMINED GAMES

We turn now from linear programming to the theory of games of strategy. Ultimately these two theories can be closely connected but superficially they are different.

Game theory considers situations in which there are two (or more) persons, each of whose actions influence, but do not completely determine, the outcome of a certain event. The manner in which their actions influence the outcome of the event is spelled out in the rules of play of the game. Depending on which event actually occurs, the players receive various payments, which we shall assume to be in money. If, for each possible event, the algebraic sum of payments to all players is zero, the game is called *zero-sum;* otherwise it is *nonzero-sum.* Usually the players will not agree as to which event should occur, so that their objectives in the game are different. In the case of a matrix game, which is a two-person game in which one player loses what the other wins (i.e., a two-person zero-sum game), game theory provides a solution, based on the principle that each player tries to choose his course of action so that, regardless of what his opponent does, he can assure himself of at least a certain amount.

Most recreational games such as tick-tack-toe, checkers, backgammon, chess, poker, bridge, and other card games can be viewed as games of strategy. Moreover, they can be put in the form of matrix games; the way in which this is done will be discussed for specific examples in Sections 6 and 9. On the other hand, gambling games such as dice, roulette, etc., are *not* (as usually formulated) games of strategy, since a person playing one of these games is merely "betting against the odds."

The actual games of strategy mentioned above are nearly all too complicated, as they stand, to be analyzed completely. We shall instead construct simple examples which, although uninteresting from a player's point of view, do illustrate the theory and are amenable to computations.

In this section we shall discuss strictly determined matrix games. The general definition and discussion of matrix games is given in Section 6.

Example 1. Consider the following very simple card game. There are two players, call them R and C (the reason for the use of these two letters will be explained later); player R is given a hand consisting of a red 5 and a black 5, while player C is given a black 5, a red 3, and a red 1. The game they are to play is the following: At a given signal the players *simultaneously* expose one of their cards. If the cards *match* in color, player R wins the (positive) difference between the numbers on the cards; if the cards do *not match* in color, player C wins the (positive) difference between the numbers on the cards played. Obviously the strategic decision that each player must make is which of his cards to play.

		Player C		
		bk 5	rd 3	rd 1
Player R	bk 5	0	−2	−4
	rd 5	0	2	4

Figure 9

A convenient way of representing the game is by means of the matrix G shown in Figure 9. (In game theory it is customary to present matrices in this "table" form.) The rows represent the possible choices of player R, and the columns, the possible choices of player C; hence our use of R and C. The number in position g_{ij} represents the gain of R if R chooses row i and C chooses column j. A positive entry is a payment from C to R, while a negative "gain" for R is a payment from R to C. For instance, if R chooses row 1 (plays bk 5) and C chooses column 1 (plays bk 5), then R wins the difference of the two numbers, which is 0. If R chooses row 1 but C chooses column 2 (plays rd 3), then C wins the difference of 5 minus 3, which is indicated by the −2 entry in the matrix. The rest of the entries are determined similarly.

The game shown in Figure 9 is called a *matrix game*. Any matrix can be considered a two person matrix game by allowing one player to control the rows, the other the columns, and defining the payoffs of the game to be the various matrix entries. In Section 6 such games will be discussed in detail.

How should the players play the matrix game of Figure 9? Player C would like to get the −4 entry in the matrix. However, the only way he could get it would be to play the third column of the matrix, in which case player R would surely choose the second row and C would lose 4 rather than gain 4. On the other hand, if C chooses the first column (i.e., plays bk 5), he assures himself that he will break even regardless of what R does. It is clear that R has nothing to lose and may possibly gain by choosing the second row, hence he should always do so. The knowledge that he will do so reinforces C in his choice of the first column. The optimal procedure for the players is then: R should play rd 5 and C should play bk 5. If they play this way, neither player wins from the other, that is, the game is *fair*.

A command of the form: "Play rd 5," or "Play bk 3," will be called a *strategy*. If player R uses the strategy "Play rd 5" in the game of Figure 9, then, regardless of what C does, R assures himself that he will get *at least* a payoff of zero. Similarly, if C uses the strategy "Play bk 5," then, regardless of what R does, C assures himself of obtaining a payoff of *at most zero*, i.e., a loss of at most zero. Since R cannot, by his own efforts, assure himself of gaining more than zero, and C cannot, by his own efforts, assure himself of losing less than zero, and since these two numbers are the same, we call these *optimal strategies* for the game. Also we call zero the *value* of the game, since it is the outcome of the game if each player uses his optimal strategy.

DEFINITION. We shall say that a matrix game is *strictly determined* if the matrix contains an entry, call it v, which is *simultaneously* the *minimum* of the row in which it occurs and the *maximum* of the column in which it occurs. *Optimal strategies* for the players are then the following.

For player R: "Play a row that contains v."
For player C: "Play a column that contains v."

The *value* of the game is v. The game is *fair* if its value is zero.

In Section 6 it will be shown that the strategies here defined are optimal in the sense indicated above, and that v has the property of being the best either player can assure for himself.

Example 1 (continued). The game of Figure 9 is strictly determined, since the 0 entry in the lower left-hand corner of the matrix is the minimum of the second row and the maximum of the first column of that matrix. Observe that the optimal strategies given in the definition above agree with those found earlier. Also the value of that game is zero, according to the above definition; hence it is fair.

Example 2. Another example of a strictly determined matrix game is shown in Figure 10. Note that the two 2 entries in the second row each are the minimum of the row and maximum of the column in which they occur. Hence the value of the game is 2 and optimal strategies are: for R, choose row 2 always; for C, choose either column 2 or column 4.

−7	0	12	−1
4	2	7	2
−3	−1	5	0

Figure 10

The solution of a strictly determined game is particularly easy to find since each player can calculate the other's optimal strategy and hence know what he will do. Not all matrix games are so easy to solve, as we shall see in the next section.

In Figure 11 we show three matrix games. The game in Figure 11a

(a)

0	1
−3	10

(b)

5	2
−7	−4

(c)

0	1
2	0

Figure 11

is strictly determined and fair, and its optimal strategies are for R to choose the first row and C to choose the first column. The game in Figure 11b is strictly determined but not fair, since its value is 2. What are its optimal strategies? Finally, the game in Figure 11c is not strictly determined, and the solution of games such as this one will be the subject of the next section.

EXERCISES

1. Determine which of the games given below are strictly determined and which are fair. When the game is strictly determined, find optimal strategies for each player.

(a)

0	2
−1	4

(b)

5	0
0	2

(c)

3	1
4	0

(d)

1	−1
−1	1

(e)

3	1
−4	0

(f)

0	4
0	2

(g)

7	0
0	0

(h)

0	0
0	−7

(i)

0	0
0	0

(j)

1	1
1	1

[*Ans.* (a) Strictly determined and fair; R play row 1, C play column 1; (b) nonstrictly determined; (e) strictly determined but not fair; R play row 1, C play column 1; (j) strictly determined but not fair; both players can use any strategy.]

2. In Example 1, suppose that R is given rd 5 and bk 3, and C is given bk 3 and rd 3. Set up the matrix game corresponding to it. Is it strictly determined? Is it fair? Find optimal strategies for each player.

[*Ans.* Yes; yes; both play bk 3.]

3. Each of the two players shows one or two fingers (simultaneously) and C pays to R a sum equal to the total number of fingers shown. Write the game matrix. Show that the game is strictly determined, and find the value and optimal strategies.

4. Each of two players shows one or two fingers (simultaneously) and C pays to R an amount equal to the total number of fingers shown, while R pays to C an amount equal to the product of the numbers of fingers shown. Construct the game matrix (the entries will be the net gain of R), and find the value and the optimal strategies.

[*Ans.* $v = 1$, R must show one finger, C may show one or two.]

5. Show that a strictly determined game is fair if and only if there is a zero entry such that all entries in its row are nonnegative and all entries in its column are nonpositive.

6. Consider the game

$$G = \begin{bmatrix} 2 & 5 \\ -1 & a \end{bmatrix}.$$

(a) Show that G is strictly determined regardless of the value of a.
(b) Find the value of G. [*Ans.* 2.]
(c) Find optimal strategies for each player.
(d) If $a = 1{,}000{,}000$, obviously R would like to get it as his payoff. Is there any way he can assure himself of obtaining it? What would happen to him if he tried to obtain it?
(e) Show that the value of the game is the most that R can assure for himself.

7. Consider the matrix game

$$G = \begin{bmatrix} a & a \\ c & d \end{bmatrix}.$$

show that G is strictly determined for every set of values for a, c, and d. Show that the same result is true if two entries in a given column are always equal.

8. Find necessary and sufficient conditions that the game

$$G = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

should be strictly determined. (*Hint:* These will be expressed in terms of relations among the numbers a and b and the number zero.)

9. Suppose that in Example 1, player R is given a hand consisting of bk x and rd y, and player C is given bk u and rd v, where x, y, u, and v are real numbers. Verify that the matrix game which they play is the following.

		Player C: bk u	Player C: rd v
Player R	bk x	$x - u$	$v - x$
	rd y	$u - y$	$y - v$

(a) Show that if $x = u$, $v \geq x$, and $y \geq x$, the game is strictly determined and fair.
(b) Show that if $y = v$, $y \leq x$, and $y \leq u$, the game is strictly determined and fair.

10. Consider a strictly determined 2×2 matrix game G. Suppose u and v are two entries of the matrix such that each is the minimum of the row and the maximum of the column in which it occurs. Show that $u = v$.

SUPPLEMENTARY EXERCISES

11. In Example 1 assume that R has bk 5 and bk 4, and C has bk 4 and rd 2. Show that the game is favorable to C, and find optimal strategies.

[*Ans.* R choose bk 4; C choose rd 2; $v = -2$.]

12. In Example 1 assume that R has bk a and bk b, while C has bk c and rd d. If $a \geq b \geq c \geq d$, show that the game is always strictly determined. Do the same if the inequalities are reversed.

13. Solve the following games.

(a)

1	5	1	7
−2	8	0	−9
1	12	1	3

(b)

1	−12	6
0	−4	1
3	−7	2
3	−4	2
−5	−4	7

[*Ans.* R play either row 2 or 4; C play column 2; $v = -4$.]

14. Show that the following game is always strictly determined for non-negative a and any values of the parameters b, c, d, and e.

$2a$	a	$3a$
b	$-a$	c
d	$-2a$	e

.

15. For what values of a is the following game strictly determined?

a	6	2
-1	a	-7
-2	4	a

.

[*Ans.* $-1 \leq a \leq 2$.]

5. *NONSTRICTLY DETERMINED GAMES*

As we saw in the numerical examples of the last section, some matrix games are nonstrictly determined, that is, they have no entry which is simultaneously a row minimum and a column maximum. We can characterize nonstrictly determined 2×2 matrix games as follows.

Theorem. The matrix game

$G =$

a	b
c	d

is nonstrictly determined if and only if the two entries on one of the diagonals are both *larger* than the two entries on the other diagonal; that is, a and d are both larger or both smaller than b and c.

Proof. If the two entries on one of the diagonals are both larger than the two entries on the other diagonal, then it is easy to check that no entry of the matrix is simultaneously the minimum of the row and the maximum of the column in which it occurs; hence the game is not strictly determined.

To prove the other half of the theorem suppose that the game is nonstrictly determined, and suppose that the rows and columns have been arranged so that a is the largest entry in the matrix. Then, by Exercise 7 of the preceding section, no two entries in the same row or the same column are equal, since that means that the game is strictly determined. Hence, a is larger than both b and c. We must now show that d is larger than both b and c. First, d must be larger than b, for if it were less, b would be the minimum of the first row and the maximum of the second column and the game would be strictly determined. Second, d is larger than c, for if it were less, d would be the minimum of the second row and the maximum of the second column, and the game would be strictly determined. This completes the proof of the theorem.

Example 1. Consider the card game of the example in the last section and assume that player R has bk 5 and rd 3 while player C has bk 3 and rd 5. The rules of play are as before. The corresponding matrix game is

Player C

		bk 3	rd 5
Player R	bk 5	2	0
	rd 3	0	2

which clearly is nonstrictly determined.

Example 2. Consider the following game played by two people, Jones and Smith. Jones conceals either a \$1 or a \$2 bill in his hand; Smith guesses 1 or 2, and wins the bill if he guesses its number. The matrix of this game is

Smith guesses

		1	2
Jones chooses	\$1 bill	−1	0
	\$2 bill	0	−2

Again the game is nonstrictly determined.

How should one play a nonstrictly determined game? We must first convince ourselves that no one choice is clearly optimal for either player. In Example 1, R would like to win 2. But if he definitely chooses bk 5, and C finds this out, C can bring about a zero by playing rd 5. If R chooses rd 3, C can bring about a zero by playing bk 3. Similarly, if C's choice is found out by R, then R can win 2. So our first result is that each player must, in some way, prevent the other player from finding out which card he is going to play.

We also note that for a single play of the game there is no difference between the two strategies, as long as one's strategy is not guessed by the opponent. Let us now consider the game being played several times. What should R do? Clearly, he should not play the same card all the time, or C will be able to notice what R is doing, and profit by it. Rather, R should sometimes play one card, and sometimes the other! Our key question then is, "How often should R play each of his cards?" From the symmetry of the problem we can guess that he should play each card as often as the other, hence each one-half the time. (We will see later that this is, indeed, optimal.) In what order should he do this? For example, should he alternate bk 5 and rd 3? That is dangerous, because if C notices the pattern, he will gain by knowing just what R will do next. Thus we see that R should play bk 5 half the time, but according to some unguessable pattern. The only safe way of doing this is to play it half the time at random. He could, for example, toss a coin (without letting C see it) and play bk 5 if it comes up heads, rd 3 if it comes up tails. Then his opponent cannot guess his decision, since he himself won't know what the decision is. Thus we conclude that a rational way of playing is for each player to *mix* his strategies, selecting sometimes one, sometimes the other; and these strategies should be selected at random, according to certain fixed ratios (probabilities) of selecting each.

By a *mixed strategy* for player R we shall mean a command of the form, "Play row 1 with probability p_1 and play row 2 with probability p_2," where we assume that $p_1 \geq 0$ and $p_2 \geq 0$ and $p_1 + p_2 = 1$. Similarly, a mixed strategy for player C is a command of the form, "Play column 1 with probability q_1 and play column 2 with probability q_2," where $q_1 \geq 0$, $q_2 \geq 0$, and $q_1 + q_2 = 1$. A mixed strategy vector for player R is the probability row vector (p_1, p_2), and a mixed strategy vector for player C is the probability column vector $\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$.

Examples of mixed strategies are $(\frac{1}{2}, \frac{1}{2})$ and $\begin{pmatrix} \frac{1}{5} \\ \frac{4}{5} \end{pmatrix}$. The reader may wonder how a player could actually play one of these strategies. The mixed strategy $(\frac{1}{2}, \frac{1}{2})$ is easy to realize since it is simply the coin-flipping strategy described above. The mixed strategy $\begin{pmatrix} \frac{1}{5} \\ \frac{4}{5} \end{pmatrix}$ is more difficult to realize since there is no chance device in common use that gives these probabilities. However, suppose that a pointer is constructed with a card that is $\frac{4}{5}$ shaded and $\frac{1}{5}$ unshaded, as in Figure 12, and C simply spins the pointer (without letting R see it, of course!). Then, if the pointer stops on the unshaded part, he plays the first column, and if it stops on the shaded part, he plays the second column, and thus realizes the desired strategy. By varying the proportion of shaded area on the card other mixed strategies can conveniently be realized.

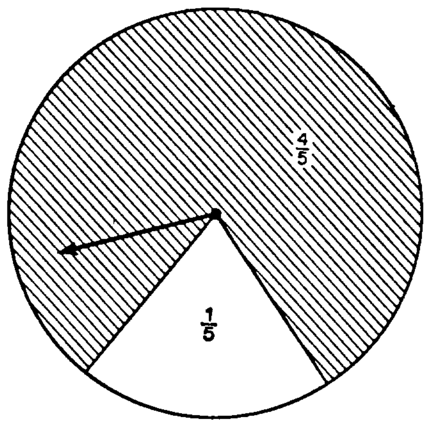


Figure 12

Consider the nonstrictly determined game

$$G = \begin{array}{|c|c|} \hline a & b \\ \hline c & d \\ \hline \end{array}\,.$$

Having argued, as above, that the players should use mixed strategies in playing a nonstrictly determined game, it is still necessary to decide how to choose an optimal mixed strategy.

DEFINITION. For the nonstrictly determined game G the number v is its *value* and $p^0 = (p_1^0, p_2^0)$ and $q^0 = (q_1^0, q_2^0)$ are *optimal strategies* for R and C, respectively, if the following inequalities are satisfied.

(1) $$p^0G = (p_1^0, p_2^0)\begin{pmatrix} a & b \\ c & d \end{pmatrix} \geq (v, v)$$

(2) $$Gq^0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} q_1^0 \\ q_2^0 \end{pmatrix} \leq \begin{pmatrix} v \\ v \end{pmatrix}.$$

(If z and w are vectors, the inequality $z \geq w$ means that each component of z is greater than or equal to the corresponding component of w.) The game is *fair* if $v = 0$.

If R chooses a mixed strategy $p = (p_1, p_2)$ and (independently) C chooses a mixed strategy $q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$, then player R obtains the payoff a with probability p_1q_1; he obtains the payoff b with probability p_1q_2; he obtains c with probability p_2q_1; and he obtains d with probability p_2q_2; hence his mathematical expectation (see Chapter IV, Section 12) is then given by the expression

$$ap_1q_1 + bp_1q_2 + cp_2q_1 + dp_2q_2 = pGq.$$

By a similar computation, one can show that player C's expectation is the negative of this expression.

To justify this definition we must show that if v, p^0, q^0 exist for G, each player can guarantee himself an expectation of v. Let q be any strategy for C. Multiplying (1) on the right by q, we get $p^0Gq \geq (v, v)q = v$, which shows that, regardless of how C plays, R can assure himself of an expectation of at least v. Similarly, let p be any strategy vector for R. Multiplying (2) on the left by p, we obtain $pGq^0 \leq p\begin{pmatrix} v \\ v \end{pmatrix} = v$, which shows that, regardless of how R plays, C can assure himself of an expectation of at most v. It is in this sense that p^0 and q^0 are optimal. It follows further that, if both players play optimally, then R's expectation is exactly v and C's expectation is exactly v. (Compare Exercise 11.)

We must now see whether there are strategies p^0 and q^0 for the game G. While in more complicated games the finding of optimal strategies is a difficult task, for a 2×2 nonstrictly determined game the following formulas provide the solution.

(3) $$p_1^0 = \frac{d - c}{a + d - b - c}$$

(4) $$p_2^0 = \frac{a - b}{a + d - b - c}$$

(5) $$q_1^0 = \frac{d - b}{a + d - b - c}$$

(6) $$q_2^0 = \frac{a - c}{a + d - b - c}$$

(7) $$v = \frac{ad - bc}{a + d - b - c}.$$

It is an easy matter to verify (see Exercise 12) that formulas (3)–(7) satisfy conditions (1)–(2). Actually, the inequalities in (1) and (2) become equalities in this simple case, a fact that is not true in general for nonstrictly determined games of larger size.

Formulas (3)–(7) look rather complicated, but the mnemonic device of Figure 13 will make it unnecessary to remember them in detail. Observe that the numerators of (3)–(6) are the differences of the main

	$d - b$	$a - c$
$d - c$	a	b
$a - b$	c	d

Figure 13

diagonal and the other diagonal entries. We take the difference of the entry on the main diagonal and the other entry in the first row, and write it in the second row. Then we take the corresponding difference in the second row, and write it in the first row. For the column player we do the same, substituting "column" for "row." In order to convert these differences into strategies, simply divide by the sum of the differences, which is the denominator of each of the expressions (3)–(6). Finally, to find the value of the game, multiply one of the optimal strategy vectors into the game matrix. Let us illustrate these ideas on the two examples discussed earlier.

Example 1 (continued). Applying the rules of Figure 13, we have

	$d - b = 2$	$a - c = 2$
$d - c = 2$	2	0
$a - b = 2$	0	2

The sum $a + d - b - c = 2 + 2 = 4$, so that the optimal strategies are $(\frac{1}{2}, \frac{1}{2})$ for R and $\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$ for C. Hence each player should use the coin-flipping strategy for optimal results. The value of the game is obtained by multiplying R's optimal strategy into the first column of the matrix, $v = (\frac{1}{2}, \frac{1}{2})\begin{pmatrix}2\\0\end{pmatrix} = 1$. Thus the game is biased in R's favor, and he has an expected gain of \$1 per game.

Example 2 (continued). Again applying the rules of Figure 13,

	$d - b = -2$	$a - c = -1$
$d - c = -2$	-1	0
$a - b = -1$	0	-2

The sum is $a + d - b - c = -3$, so that the optimal strategies are $(\frac{2}{3}, \frac{1}{3})$ for R and $\begin{pmatrix}\frac{2}{3}\\\frac{1}{3}\end{pmatrix}$ for C. The value of the game is $(\frac{2}{3}, \frac{1}{3})\begin{pmatrix}-1\\0\end{pmatrix} = -\frac{2}{3}$, which means that the game is biased in Smith's favor. Smith should then pay $66\frac{2}{3}$ cents to play the game, in order to make it fair, that is, to make its value zero.

Example 3. As one final example, consider the following matrix game.

6	2
1	4

Applying the rules of Figure 13, we obtain

	2	5
3	6	2
4	1	4

which means that optimal strategies are $(\frac{3}{7}, \frac{4}{7})$ for R and $\begin{pmatrix}\frac{2}{7}\\\frac{5}{7}\end{pmatrix}$ for C. The value of the game is $(\frac{3}{7}, \frac{4}{7})\begin{pmatrix}6\\1\end{pmatrix} = \frac{22}{7}$.

EXERCISES

1. Find the optimal strategies for each player and the values of the following games.

(a)

1	2
3	4

(b)

1	0
−1	2

(c)

2	3
1	4

(d)

15	3
−1	2

(e)

7	−6
5	8

(f)

3	15
−1	10

[*Ans.* (a) $v = 3; (0, 1); \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. (b) $v = \frac{1}{2}; (\frac{3}{4}, \frac{1}{4}); \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$.

(d) $v = 3; (1, 0); \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. (e) $v = \frac{43}{8}; (\frac{3}{16}, \frac{13}{16}); \begin{pmatrix} \frac{7}{8} \\ \frac{1}{8} \end{pmatrix}$.]

2. Set up the ordinary game of matching pennies as a matrix game. Find its value and optimal strategies. How are the optimal strategies realized in practice by players of this game?

3. A version of two-finger Morra is played as follows: Each player holds up either one or two fingers; if the sum of the number of fingers shown is even, player R gets the sum, and if the sum is odd, player C gets it.

(a) Show that the game matrix is

		Player C 1	Player C 2
Player R	1	2	−3
	2	−3	4

(b) Find optimal strategies for each player and the value of the game.

[*Ans.* $(\frac{7}{12}, \frac{5}{12})$; $v = -\frac{1}{12}$.]

4. Rework Exercise 3 if player C gets the even sum and player R gets the odd sum.

5. Consider the following "war" problem: Some attacking bombers are attempting to bomb a city that is protected by fighters. The bombers can each day attack either "high" or "low," the low attack making the bombing more accurate. Similarly, the fighters can each day look for the bombers either "high" or "low." Credit the bombers with six points if they avoid the fighters, and zero if the fighters find them. Also credit the bombers with three extra points for accurate bombing if they fly low.

(a) Set up the game matrix.
(b) Find optimal strategies for each player.
(c) Give instructions to the bomber and fighter commanders so that by flipping coins they can decide what to do.

[*Ans.* (c) The bomber commander should flip one coin to decide whether to go high or low. The fighter commander should flip two coins and then go high if both turn up heads.]

6. Generalize the problem in Exercise 5 by crediting the bombers with x points for avoiding the fighters and y points for flying low. (Assume that x and y are positive.)

(a) Set up the matrix.
(b) If $y \geq x$, show that the game is strictly determined and find optimal strategies.
(c) If $y < x$, show that the game is nonstrictly determined and find optimal strategies.
(d) Comment on these results, with special attention to the bombers' strategies.

7. If $G =$

a	b
c	d

is nonstrictly determined, prove that it is fair if and only if

$$ad - bc = 0.$$

8. In formulas (3)–(7) prove that $p_1 > 0$, $p_2 > 0$, $q_1 > 0$, and $q_2 > 0$. Must v be greater than zero?

9. Utilizing the results of Exercise 7 of the last section, find necessary and sufficient conditions that the game

$G =$

a	0
0	b

be nonstrictly determined. Find optimal strategies for each player and the value of G, if it is nonstrictly determined.

[*Ans.* a and b must be both positive or both negative.

$$p_1 = \frac{b}{a+b};\ p_2 = \frac{a}{a+b};\ q_1 = \frac{b}{a+b};\ q_2 = \frac{a}{a+b};\ v = \frac{ab}{a+b}.]$$

10. Suppose that R is given bk x and rd y while C is given bk u and rd v (where x, y, u, and v stand for positive integers). Let them play the matrix game

	bk u	rd v
bk x	xu	$-xv$
rd y	$-yu$	yv

Show that the game is always nonstrictly determined and always fair.

11. If G, p^0, q^0, and v are as in the definition, show that $v = p^0Gq^0$.

12. Verify that (3)–(7) satisfy the conditions (1) and (2).

SUPPLEMENTARY EXERCISES

The remaining exercises refer to a special case of the *product payoff game* (due to A. W. Tucker, see Exercises 14–20 of Section 7). In this game, R is given two numbers x and y such that $xy < 0$, and C is given two numbers u and z such that $uz < 0$. They play the game whose matrix is

	u	z
x	xu	xz
y	yu	yz

In other words, each player chooses one of his two numbers, and they then exchange a sum of money equal to the product of these numbers. Since $xy < 0$ and $uz < 0$, each player must have one negative and one positive number.

13. Assume that $x > 0 > y$ and $u > 0 > z$. Show that the game is always nonstrictly determined.

14. Show that we can always assume that x, y, u, and z satisfy the relationships given in Exercise 13 by, if necessary, relabelling and reordering rows and columns.

15. Show that the game is always fair. [*Hint:* Use Exercise 7.]

16. For the situation in Exercise 13, find optimal strategies for both players. [*Partial Ans.* For R the optimal strategy is $(-y/(x - y), x/(x - y))$.]

17. Solve the game for $x = 3$, $y = -5$, $u = 4$, and $z = -2$.
[*Partial Ans.* For R the answer is $(\frac{5}{8}, \frac{3}{8})$ and $v = 0$.]

6. *MATRIX GAMES*

We shall consider a large class of games in this section, and discuss them in considerable generality. Our games are played between two players, according to strictly specified rules. Each player performs certain actions, as specified by the rules of the game, and then, at the end of the play of the game, one of the players may have to pay a sum of money to the other player. The game may be repeated many times.

During such a game a player may have to make many strategic decisions. By a (pure) *strategy* for one of the players we mean a complete set of rules as to how he should make his decisions. We shall illustrate this in terms of the game of tick-tack-toe (and nearly the same remarks would apply to any game in which the players take turns moving). Let us construct a strategy for the player who moves first. His first decision concerns the opening move. He may choose any one of nine squares, and the strategy must tell him which choice to make. Let us say we tell him to move into the upper left-hand corner. His opponent may answer this in one of eight ways, and the strategy must be prepared for each alternative. It must have eight rules, such as "If he moves into the middle, move into the lower right-hand corner!" For every such move the opponent may respond with one of several alternatives, and the strategy must again have an answering move ready for each of them, etc. Hence the strategy takes into account every conceivable position of the first player, and instructs what move to make in each one.

A strategy may be thought of as a set of instructions to be given to a machine, so that the machine will play the game exactly the way we would have.

We number the strategies of the first player $1, 2, \ldots, m$, and those of the second player $1, 2, \ldots, n$. Since each of the players must play according to one of his strategies, the game may proceed in any one of mn ways, and if each player chooses a definite strategy, the outcome is determined. We may think of giving the two strategies to two machines, and let them work out what happens. Let us suppose that, when the first player chooses strategy i and the second strategy j, the former wins an amount a_{ij}. We arrange these numbers a_{ij} into an $m \times n$ matrix, the *game matrix*. We may then think of the game as consisting of a choice of a row by the first player, and a column by the second player. Hence we see that any game specified by rules may be thought of as a matrix game.

Conversely, every matrix can be considered as a game. An $m \times n$ matrix may be thought of as a game between two players, in which player R chooses one of the m rows and player C simultaneously chooses one of the n columns. The outcome of the game is that C pays to R an amount equal to the entry of the matrix in the chosen row and column. (A negative entry represents a payment from R to C, as usual.)

In an $m \times n$ matrix game, the player R has m pure strategies, and the player C has n. We have seen in the last section that, in addition, we must consider the mixed strategies of the two players. We extend this concept to $m \times n$ games.

DEFINITION. An m-component row vector p is a *mixed-strategy vector* for R if it is a probability vector; similarly, an n-component column vector q is a mixed-strategy vector for C if it is a probability vector. (Recall from Chapter V that a probability vector is one with nonnegative entries whose sum is 1.) Let V and V' be the vectors

$$V = \underbrace{(v, v, \ldots, v)}_{m \text{ components}} \quad \text{and} \quad V' = \left.\begin{pmatrix} v \\ v \\ \cdot \\ \cdot \\ \cdot \\ v \end{pmatrix}\right\} n \text{ components},$$

where v is a number. Then v is the *value of the game* and p^0 and q^0 are *optimal strategies* for the players if and only if the following inequalities hold.

$$p^0G \geq V$$
$$Gq^0 \leq V'.$$

In Sections 4 and 5 we have given several examples of such matrix games together with their solutions. Notice that we have *not* proved that an arbitrary matrix game has a value and optimal strategies for each player; that question will be discussed later.

Theorem 1. If G is a matrix game which has a value and optimal strategies, then the value of the game is unique.

Proof. Suppose that v and w are two different values for the game G. Let $V = (v, v, \ldots, v)$ and $W = (w, w, \ldots, w)$ be m-component row vectors, and let

$$V' = \begin{pmatrix} v \\ v \\ \cdot \\ \cdot \\ \cdot \\ v \end{pmatrix} \quad \text{and} \quad W' = \begin{pmatrix} w \\ w \\ \cdot \\ \cdot \\ \cdot \\ w \end{pmatrix}$$

be n-component column vectors. Then let p^0 and q^0 be optimal mixed strategy vectors associated with the value v so that

(a) $$p^0G \geq V,$$

(b) $$Gq^0 \leq V'.$$

Similarly, let p^1 and q^1 be optimal mixed strategy vectors associated with the value w so that

(c) $$p^1G \geq W,$$

(d) $$Gq^1 \leq W'.$$

If we now multiply (a) on the right by q^1, we get $p^0Gq^1 \geq Vq^1 = v$. In the same way, multiplying (d) on the left by p^0 gives $p^0Gq^1 \leq w$. The two inequalities just obtained show that $w \geq v$.

Next we multiply (b) on the left by p^1 and (c) on the right by q^0, obtaining $v \geq p^1Gq^0$ and $p^1Gq^0 \geq w$, which together imply that $v \geq w$.

Finally, we see that $v \leq w$ and $v \geq w$ imply together that $v = w$, that is, the value of the game is unique.

Theorem 2. If G is a matrix game with value v and optimal strategies p^0 and q^0, then $v = p^0Gq^0$.

Proof. By definition, v, p^0, and q^0 satisfy

$$p^0G \geq V \quad \text{and} \quad Gq^0 \leq V'.$$

Multiplying the first of these inequalities on the right by q^0, we get $p^0Gq^0 \geq v$. Similarly, multiplying the second inequality on the left by p^0, we obtain $p^0Gq^0 \leq v$. These two inequalities together imply that $v = p^0Gq^0$, concluding the proof.

Theorem 2 is important because it permits us to give an interpretation of the *value* of a game as an *expected value* in the sense of probability (see Chapter IV, Section 12). Briefly the interpretation is the following: If the game G is played repeatedly and if each time it is played player R uses the mixed strategy p^0 and player C uses the mixed strategy q^0, then the value v of G is the expected value of the game for R. The law of large numbers implies that, if the number of plays of G is sufficiently large, then the average value of R's winnings will (with high probability) be arbitrarily close to the value v of the game G.

As an example, let G be the matrix of the game of matching pennies, i.e.,

$$G = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

As was found in Exercise 2 of the last section, optimal strategies in this game are for R to choose each row with probability $\frac{1}{2}$ and for C to choose each column with probability $\frac{1}{2}$. The value of G is zero. Notice that the only two payoffs that result from a single play of the game are $+1$ and -1, neither of which is equal to the value (zero) of the game. However, if the game is played repeatedly, the average value of R's payoffs will approach zero, which is the value of the game.

Theorem 3. If G is a game with value v and optimal strategies p^0 and q^0, then v is the largest expectation that R can assure for himself. Similarly, v is the smallest expectation that C can assure for himself.

Proof. Let p be any mixed strategy vector of R and let q^0 be an optimal strategy for C; then multiply the equation $Gq^0 \leq V'$ on the left by p, obtaining $pGq^0 \leq v$. The latter equation shows that, if C plays optimally, the most that R can assure for himself is v. Now let p^0 be optimal for R; then, for every q, $p^0Gq \geq v$, so that R can actually assure himself of an expectation of v. The proof of the other statement of the theorem is similar.

Theorem 3 gives an intuitive justification to the definition of value and optimal strategies for a game. Thus the value is the "best" that a player can do and optimal strategies are the means of achieving this "best."

Matrix game theory would not be of very great interest unless we knew under what conditions such a game has a solution. The fundamental theorem of game theory is that every matrix game has a solution. The proof of this theorem is too difficult to be included here, but we do discuss its proof for the 2×2 case.

Theorem 4 (Fundamental theorem). Let G be any $m \times n$ matrix game; then there exists a value v for G and optimal strategies p^0 for player R and q^0 for player C. In other words, every matrix game possesses a solution.

Proof for 2×2 matrices. If G is strictly determined, the value and optimal strategies were found in Section 4. If G is not strictly determined, formulas (3) through (7) of Section 5 give the optimal strategies and value for G. Since G must be either strictly determined or non-strictly determined, we have covered all cases.

EXERCISES

1. Find the value and optimal strategies for the following games.

(a)

15	2	−3
6	5	7
−7	4	0

[*Ans.* $v = 5$; $(0, 1, 0)$; $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.]

(b)

5	2	−1	−1
1	1	0	1
3	0	−3	7

(c)

0	5	6	−3
1	−1	2	3
1	2	3	4
−1	0	7	5

.

2. Verify that the strategies $p^0 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and

$$q^0 = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}$$

are optimal in the game G whose matrix is

$$G = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

What is the value of the game?

3. Generalize the result of Exercise 2 to the game G whose matrix is the $n \times n$ identity matrix.

4. Suppose that player R tries to find C in one of three towns X, Y, and Z. The distance between X and Y is five miles, the distance between Y and Z is five miles, and the distance between Z and X is ten miles. Assume that R and C can go to one and only one of the three towns and that if they both go to the same town, R "catches" C and otherwise C "escapes." Credit R with ten points if he catches C, and credit C with a number of points equal to the distance he is away from R if he escapes.

(a) Set up the game matrix.
(b) Show that both players have the same optimal strategy, namely, to go to towns X and Z with equal probabilities and to go to town Y with probability $\frac{1}{4}$.
(c) Find the value of the game.

5. A version of five-finger Morra is played as follows: Each player shows from one to five fingers, and the sum is divided by three. If the sum is exactly divisible by three, there is no exchange of payoffs. If there is a remainder of one, player R wins a sum equal to the total number of fingers, while if the remainder is two, player C wins the sum.

(a) Set up the game matrix. [*Hint:* It is 5×5.]
(b) Verify that an optimal strategy for either player is to show one or five fingers with probability $\frac{1}{9}$, to show two or four fingers with probability $\frac{2}{9}$, and to show three fingers with probability $\frac{1}{3}$.
(c) Is the game fair? [*Ans.* Yes.]

6. Consider the following game:

$$G = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}.$$

(a) If a, b, and c are not all of the same sign, show that the game is strictly determined with value zero.

(b) If a, b, and c are all of the same sign, show that the vector

$$\frac{bc}{ab + bc + ca}, \quad \frac{ca}{ab + bc + ca}, \quad \frac{ab}{ab + bc + ca}$$

is an optimal strategy for player R.

(c) Find player C's optimal strategy for case (b).

(d) Find the value of the game for case (b) and show that it is positive if a, b, and c are all positive, and negative if they are all negative.

7. Two players agree to play the following game. The first player will show one, two, or four fingers. The second player will show two, three, or five fingers, simultaneously. If the sum of the fingers shown is three, five, or nine, the first player receives this sum. Otherwise no payment is made.

(a) Set up the game matrix.

(b) Use the results of Exercise 6 to solve the game.

(c) How much should the first player be willing to pay to play the game? [*Ans.* $\frac{45}{29}$.]

8. Consider the (symmetric) game whose matrix is

$$G = \begin{pmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{pmatrix}.$$

(a) If a and b are both positive or both negative, show that G is strictly determined.

(b) If b and c are both positive or both negative, show that G is strictly determined.

(c) If $a > 0$, $b < 0$, and $c > 0$, show that an optimal strategy for player R is given by

$$\frac{c}{a - b + c}, \quad \frac{-b}{a - b + c}, \quad \frac{a}{a - b + c}.$$

(d) In part (c) find an optimal strategy for player C.

(e) If $a < 0$, $b > 0$, and $c < 0$ show that the strategy given in (c) is optimal for R. What is an optimal strategy for player C?

(f) Prove that the value of the game is always zero.

9. In a well-known children's game each player says "stone" or "scissors" or "paper." If one says "stone" and the other "scissors," then the former wins a penny. Similarly, "scissors" beats "paper," and "paper" beats "stone." If the two players name the same item, then the game is a tie.

(a) Set up the game matrix.

(b) Use the results of Exercise 8 to solve the game.

10. In Exercise 9 let us suppose that the payments are different in different cases. Suppose that when "stone breaks scissors," the payment is one cent; when "scissors cut paper," the payment is two cents; and when "paper covers stone," the payment is three cents.

(a) Set up the game matrix.

(b) Use the results of Exercise 8 to solve the game.

[*Ans.* $\frac{1}{3}$ "stone," $\frac{1}{2}$ "scissors," $\frac{1}{6}$ "paper"; $v = 0$.]

SUPPLEMENTARY EXERCISES

Exercises 13–17 refer to a special case of the *exponential payoff game.* (See also Exercises 14–20 of Section 8.) To play this game we first select a number $b > 0$. Then R is given two numbers x, y such that $xy < 0$, and C is given two numbers u, z such that $uz < 0$. They play the matrix game

	u	z
x	$\pm b^{x+u}$	$\pm b^{x+z}$
y	$\pm b^{y+u}$	$\pm b^{y+z}$

where the plus payoff is exchanged if the two numbers chosen are of the same sign, and the minus payoff is exchanged if the two numbers chosen have opposite sign.

11. For $b = 2, x = 3, y = -2, u = 1$, and $z = -4$, show that the game is

16	$-\frac{1}{2}$
$-\frac{1}{2}$	$\frac{1}{64}$

12. Assume that $x > 0 > y$ and $u > 0 > z$.

(a) Show that the game matrix is

	u	z
x	b^{x+u}	$-b^{x+z}$
y	$-b^{y+u}$	b^{y+z}

(b) Show that the game is always nonstrictly determined.

(c) Show that the game is always fair.

13. Show that we can always assume that $x > 0 > y$ and $u > 0 > z$ by, if necessary, relabelling and reordering rows and columns.

14. For the situation in Exercise 12 find optimal strategies for each player. Show that these are also optimal for the three cases found in Exercise 13.

[*Partial Ans.* For R the optimal strategy is $(b^y/(b^x + b^y), b^x/(b^x + b^y))$.]

15. Solve the game in Exercise 11.

[*Partial Ans.* For R the optimal strategy is $(\frac{1}{33}, \frac{32}{33})$.]

7. MORE ON MATRIX GAMES

We recall from Section 4 that a matrix game G is *strictly determined* if there is an entry g_{ij} in G that is the minimum entry in the ith row and the maximum entry in the jth column. (By rearranging and renumbering the rows and columns of a strictly determined matrix game G we can assume that g_{11} is an entry that is the minimum of row 1 and the maximum of column 1.)

Theorem 1. If G is a strictly determined matrix game, arranged as indicated in the definition, the value of the game is $v = g_{11}$. Moreover, optimal strategies for the players are

$$p^0 = (1, 0, 0, \ldots, 0) \quad \text{and} \quad q^0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix}.$$

[These optimal strategies simply say that R should choose the row that contains the entry g_{11} (the first row) and C should choose the column that contains the entry g_{11} (the first column).]

Proof. We set $v = g_{11}$ and let p^0 and q^0 be the strategies as defined in the statement of the theorem. We have

$$\begin{aligned} p^0G &= (g_{11}, g_{12}, \ldots, g_{1n}) \\ &\geq (g_{11}, g_{11}, \ldots, g_{11}) = V, \end{aligned}$$

where we have used the fact that g_{11} was the minimum of the first row. Similarly, using the fact that g_{11} is the maximum of the first column, we have

$$Gq^0 = \begin{pmatrix} g_{11} \\ g_{21} \\ \cdot \\ \cdot \\ \cdot \\ g_{m1} \end{pmatrix} \leq \begin{pmatrix} g_{11} \\ g_{11} \\ \cdot \\ \cdot \\ \cdot \\ g_{11} \end{pmatrix} = V'.$$

From these two inequalities and the definition of a matrix game given above, we conclude that v is the value of the game and p^0 and q^0 are optimal strategies.

Theorem 2. If g_{11} and g_{ij} are two entries of G that are the minima of the rows and the maxima of the columns in which they occur, then $v = g_{11} = g_{1j} = g_{i1} = g_{ij}$.

Proof. Using the facts that g_{11} and g_{ij} are the minima of the rows and the maxima of the columns in which they occur, we see that

$$g_{ij} \geq g_{1j} \geq g_{11}, \qquad g_{ij} \leq g_{i1} \leq g_{11}.$$

(These inequalities are redundant but still true if either $i = 1$ or $j = 1$.) These two sets of inequalities imply that $g_{ij} = g_{1j} = g_{i1} = g_{11} = v$, completing the proof of the theorem.

Example 1. Although we have proved that the value of a game is unique, it may happen that a game has more than one pair of optimal strategies. For instance, let G be the game

$$G = \begin{bmatrix} 1 & 5 & 1 & 7 \\ -2 & 8 & 0 & -9 \\ 1 & 12 & 1 & 3 \end{bmatrix}.$$

Then we see that G is strictly determined with value 1, and optimal strategies are (1, 0, 0) and (0, 0, 1) for player R and

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

for player C. In the next theorem we shall see that there are still other optimal strategies for this game.

DEFINITION. Let r and s be two strategies for a player in a matrix game; then by a *convex combination* of the two strategies, we mean an expression of the form

$$ar + (1 - a)s,$$

where a is a number satisfying $0 \leq a \leq 1$.

Theorem 3. If p^0 and p^1 are two optimal strategies for R in a matrix game G then the convex combination

$$p = ap^0 + (1 - a)p^1, \qquad 0 \leq a \leq 1$$

is also an optimal strategy for R.

Similarly, if q^0 and q^1 are optimal strategies for C in G, then the convex combination

$$q = aq^0 + (1 - a)q^1, \qquad 0 \leq a \leq 1$$

is also an optimal strategy for C.

Proof. We shall prove the first statement only and leave the second as an exercise (see Exercise 3). It is easy to show that p is a probability vector. By hypothesis, we have $p^0G \geq V$ and $p^1G \geq V$. Hence we see that

$$\begin{aligned} pG &= [ap^0 + (1 - a)p^1]G \\ &= ap^0G + (1 - a)p^1G \\ &\geq aV + (1 - a)V = V, \end{aligned}$$

which shows that p is also an optimal strategy, completing the proof of the theorem.

Example 1 (continued). Theorem 3 implies that, in Example 1, convex combinations of strategies of the form $a(1, 0, 0) + (1 - a)(0, 0, 1) = (a, 0, 1 - a)$ are optimal for R. It is easy to check that $(\frac{1}{2}, 0, \frac{1}{2})$ and $(\frac{1}{4}, 0, \frac{3}{4})$ are optimal and of this form. By similar reasoning, all strategies of the form

$$a\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + (1 - a)\begin{pmatrix}0\\0\\1\\0\end{pmatrix} = \begin{pmatrix}a\\0\\1 - a\\0\end{pmatrix},$$

for $0 \leq a \leq 1$, are optimal for C.

Theorem 4. If k is a nonnegative number, i.e., $k \geq 0$, and G is a matrix game with value v, then the game kG is a matrix game with value kv, and every strategy optimal in G is also optimal in kG. (Recall that the matrix kG is obtained from G by multiplying every entry of G by the number k.)

Proof. Let p^0 be an optimal strategy for R in the game G, that is, $p^0G \geq V$. Then we have

$$p^0(kG) = k(p^0G) \geq kV.$$

Similarly, if q^0 is optimal for C in the game G, then

$$(kG)q^0 = k(Gq^0) \leq kV'.$$

These two inequalities show that kv is the value of kG and also that optimal strategies in G are also optimal in the game kG.

It should be observed that it was essential for the proof of this theorem that k be nonnegative, since multiplying an inequality by a *negative* number has the effect of reversing the direction of the inequality sign. The following example shows that the above theorem is false for negative k's.

Example 2. Let $k = -1$ and let G and $(-1)G$ be the matrices

$$G = \begin{array}{|c|c|}\hline 2 & 3 \\ \hline -1 & 0 \\ \hline\end{array} \quad \text{and} \quad (-1)G = \begin{array}{|c|c|}\hline -2 & -3 \\ \hline 1 & 0 \\ \hline\end{array}.$$

Observe that each of these games is strictly determined but that the value of the first game is 2, while the value of the second is 0 [which is not equal to $(-1)2 = -2$]. Moreover, optimal strategies in G are for R to play the first row with probability 1, and for C to play the first column with probability 1, but neither of these strategies is optimal in the game $(-1)G$.

Theorem 5. Let G be an $m \times n$ matrix game with value v; let E be the $m \times n$ matrix each of whose entries is 1; and let k be *any* constant. Then the game $G + kE$ has value $v + k$, and every strategy optimal in the game G is also optimal in the game $G + kE$. (The game $G + kE$ is obtained from the game G by adding the number k to each entry in G.)

Proof. Let p^0 and q^0 be optimal strategies in G; then $p^0G \geq V$ and $Gq^0 \leq V'$. We have

$$\begin{aligned} p^0(G + kE) &= p^0G + p^0(kE) \\ &= p^0G + k(p^0E) \\ &\geq (v, v, \ldots, v) + (k, k, \ldots, k) \\ &= (v + k, v + k, \ldots, v + k). \end{aligned}$$

Similarly, we have

$$(G + kE)q^0 = Gq^0 + k(Eq^0)$$

$$\leq \begin{pmatrix} v \\ v \\ \cdot \\ \cdot \\ \cdot \\ v \end{pmatrix} + \begin{pmatrix} k \\ k \\ \cdot \\ \cdot \\ \cdot \\ k \end{pmatrix} = \begin{pmatrix} v + k \\ v + k \\ \cdot \\ \cdot \\ \cdot \\ v + k \end{pmatrix}.$$

These inequalities show that the value of the game $G + kE$ is $v + k$ and also show that each strategy optimal in G is optimal in $G + kE$.

Theorem 6. Let G be an $m \times n$ matrix game with value v; then there exist $k \geq 0$ and $M > 0$ so that the game $\frac{1}{M}(G + kE)$ has the same optimal strategies as G and has all its entries between 0 and 1.

Proof. Let k be the absolute value of the most negative entry in G, or 0 in case there are no negative entries. Then by Theorem 5 the game $G + kE$ has the same optimal strategies as G. By construction it is clear that $G + kE$ has all entries ≥ 0. Now let M be the maximum positive entry in $G + kE$, or 1 in case there are no positive entries. Hence $M > 0$. By Theorems 4 and 5 the game $\frac{1}{M}(G + kE)$ has the same optimal strategies as G, and by the choice of M all entries lie between 0 and 1.

The last three theorems show that the actual units used to measure the game payoffs are irrelevant as far as optimal strategies go. The only thing that is important for them is the relative magnitudes of the payoffs.

EXERCISES

1. Find the value of and all optimal strategies for the following games.

(a)

5	10	6	5
5	7	8	5
0	5	6	5

(b)

−2	0	−1
−5	7	8

(c)

0	0	1	0
1	0	0	0
1	0	1	0

(d)

3	2	3
6	2	7
5	1	4

.

[*Ans.* (a) $v = 5, (a, 1 - a, 0), \begin{pmatrix} a \\ 0 \\ 0 \\ 1 - a \end{pmatrix}$; (d) $v = 2, (a, 1 - a, 0), \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.]

2. Let G be a strictly determined game with value v. Let h be the number of rows which R can choose as an optimal strategy, and let k be the number of columns which C can choose as an optimal strategy. Prove that v occurs at least $h \cdot k$ times in G. Can it occur more than this number of times?

[*Ans.* Yes.]

3. If q^0 and q^1 are optimal strategies for C in the matrix game G, show that the strategy

$$q = aq^0 + (1 - a)q^1,$$

where a is a constant with $0 \leq a \leq 1$, is also optimal in the game G.

4. Find the values of the games kG and $G + kE$ for each of the games G whose matrices are given in Exercise 1 of Section 6, if k takes on the values 3, 0, and -2.

5. If G is any matrix game and $k = 0$, find all optimal strategies for each player in the game kG. [*Ans.* Any strategy is optimal.]

6. If G is any matrix game and $k > 0$, show that every strategy optimal in kG is also optimal in G. [*Hint:* Multiply by $1/k$.]

7. If G is any matrix game and k is any constant, show that every strategy optimal in the game $G + kE$ is also optimal in the game G.

8. Suppose that before C and R play a matrix game G, player C gives to player R a payment of k dollars. In this case we shall say C has made a *side payment* of k to R. (If k is negative, then, as usual, this will be a side payment of R to C.)

(a) If C has made a side payment of k to R before playing the game G, show that the game they actually play is $G + kE$.
(b) If v is the value of the game G, find the value of the game $G - vE$.
(c) Using the results of (a) and (b), show that any matrix game G with value v can be made into a fair matrix game by requiring that C make a side payment of $-v$ to R before they play the game G.

9. Show that any matrix game G can be made into a fair matrix game, with each entry in the matrix lying between -1 and 1, by adding the same number to each entry in the matrix and by multiplying each entry by a positive number.

10. Show that the sets of optimal strategies for each player are unchanged by the transformation suggested in Exercise 9. How does the value of the game change?

11. Consider the matrix game

a	b	b
b	a	b
b	b	a

, where $a > b$.

(a) Show that this can be obtained from the identity matrix by multiplying it by a suitable number, and then adding bE.

(b) Use the results of Section 6, Exercise 2, to solve the game.

[*Ans.* $v = (a/3) + (2b/3)$.]

12. Suppose that the entries of a matrix game are rewritten in new units (e.g., dollars instead of cents). Show that the monetary value of the game has not changed.

13. Consider the game of matching pennies whose matrix is

1	-1
-1	1

.

If the entries of the matrix represent gains or losses of one penny, would you be willing to play the game at least once? If the entries represent gains or losses of one dollar, would you be willing to play the game at least once? If they represent gains or losses of one million dollars would you play the game at least once? In each of these cases show that the value is zero and optimal strategies are the same. Discuss the practical application of the theory of games in the light of this example.

SUPPLEMENTARY EXERCISES

The remaining exercises refer to the *product payoff game* (due to A. W. Tucker). Two sets, S and T, are given, each set containing at least one positive and at least one negative number (but no zeros). Player R selects a number s from set S, and player C selects a number t from set T. The payoff is st.

14. Set up the game for the sets $S = \{1, -1, 2\}$ and $T = \{1, -3, 2, -4\}$.

[*Ans.*

1	−3	2	−4
−1	3	−2	4
2	−6	4	−8

.]

15. Consider the following mixed strategy for either player: "Choose a positive number p and a negative number n with probabilities $-n/(p - n)$ and $p/(p - n)$ respectively." Assume that R uses this strategy.
   (a) If C chooses a positive number, show that the expected payoff to R is 0.
   (b) If C chooses a negative number, show that the expected payoff to R is 0.

16. Rework Exercise 15 with R and C interchanged.

17. Use the results of Exercises 15 and 16 to show that the game is fair, and that the strategy quoted in Exercise 15 is optimal for either player.

18. Find all strategies of the kind indicated in Exercise 15 for both players for the game of Exercise 14.

[*Partial Ans.* For R they are $(\frac{1}{2}, \frac{1}{2}, 0)$ and $(0, \frac{2}{3}, \frac{1}{3})$.]

19. By subtracting ten from each entry, show that the following game is derived from a product payoff game, and find all strategies like those in Exercise 15 for both players. What is the value of the game?

11	7	12	6
9	13	8	14
12	4	14	2

[*Hint:* Use Exercises 14, 18, and Theorem 5.]

20. If a player in the product payoff game has m positive and n negative numbers in his set, show that he has mn strategies like those in Exercise 15.

8. GAMES IN WHICH ONE PLAYER HAS TWO STRATEGIES

After the 2×2 games, the simplest matrix games are the $2 \times n$ and $m \times 2$ games, i.e., where one of the players has only two strategies. Here we discuss the solution of such games.

Example 1. Suppose that Jones conceals one of the following four bills in his hand: a \$1 or a \$2 United States bill or a \$1 or a \$2 Canadian bill. Smith guesses either "United States" or "Canadian" and gets the bill if his guess is correct. The matrix of the game is the following.

			Smith Guesses: U.S.	Smith Guesses: Can.
Jones Chooses	U.S.	\$1	-1	0
	U.S.	\$2	-2	0
	Can.	\$1	0	-1
	Can.	\$2	0	-2

It is obvious that Jones should always choose the \$1 bill of either country rather than the \$2 bill, since by doing so he may cut his losses and will never increase them. This can be observed in the matrix above, since every entry in the second row is less than or equal to the corresponding entry in the first row, and every entry in the fourth row is less than or equal to the corresponding entry in the third row. In effect we can eliminate the second and fourth rows and reduce the game to the following 2×2 matrix game.

			Smith Guesses: U.S.	Smith Guesses: Can.
Jones Chooses	U.S.	\$1	-1	0
	Can.	\$1	0	-1

The new matrix game is nonstrictly determined with optimal strategies $(\frac{1}{2}, \frac{1}{2})$ for Jones and $\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$ for Smith. The value of the game is $-\frac{1}{2}$, which means that Smith should pay 50 cents to play it.

DEFINITION. Let A be an $m \times n$ matrix game. We shall say that row i *dominates* row h if every entry in row i is as large as or larger than the corresponding entry in row h. Similarly, we shall say that column j

dominates column k if every entry in column j is as small as or smaller than the corresponding entry in column k.

Any dominated row or column can be omitted from the matrix game without affecting its solution. In the original matrix of Example 1 above, we see that row 1 dominates row 2, and also that row 3 dominates row 4.

Example 2. Consider again the card game of Section 4, this time giving R a bk 5 and rd 3, while C receives a bk 6 and a bk 5 and a rd 4 and a rd 5. The matrix of the game is

	bk 6	bk 5	rd 4	rd 5
bk 5	1	0	−1	0
rd 3	−3	−2	1	2

Observe that column 3 dominates column 4; that is, C should never play rd 5. Thus our game can be reduced to the following 2 × 3 game.

	bk 6	bk 5	rd 4
bk 5	1	0	−1
rd 3	−3	−2	1

No further rows or columns can be omitted; hence we must introduce a new technique for the solution of this game. It can be shown (though we will not attempt to do so here) that, in any 2 × n game, the column player C has an optimal mixed strategy that uses only two pure strategies. Hence he may consider the game matrix two columns at a time, and select the 2 × 2 game he likes best. That is, he solves each of the 2 × 2 games consisting of two columns of the matrix, and selects the one having the smallest value.

In the above 2 × 3 game we find three games derived in this manner,

	bk 6	bk 5
bk 5	1	0
rd 3	−3	−2

	bk 6	rd 4
bk 5	1	−1
rd 3	−3	1

	bk 5	rd 4
bk 5	0	−1
rd 3	−2	1

The first game is strictly determined and fair, the second has value $-\frac{1}{3}$, and the third value $-\frac{1}{2}$. Hence player C selects the third game, i.e., he decides to use only strategies bk 5 and rd 4. The optimal strategy for the latter game is to play each card one-half of the time, hence his optimal strategy for the 2×4 game is

$$\begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}.$$

Since R knows that C will select this particular 2×2 game, R's optimal strategy is his optimal strategy in this 2×2 game, which is $(\frac{3}{4}, \frac{1}{4})$.

For an $m \times 2$ game, the row player can select which two rows to use, and he does this by selecting the 2×2 game with largest value. Then the value of the game and the optimal strategies are found by solving this 2×2 game. Similarly, for the $2 \times n$ case, C selects the two columns so that the 2×2 game resulting gives the smallest possible value (smallest loss), and then we need only solve this 2×2 game.

Example 3. A numerical example of a 3×2 game is

6	−1
0	2
4	3

Here the game is strictly determined, since the entry 3 is the minimum of its row and the maximum of its column. The value of the game is 3, and optimal strategies are $p^0 = (0, 0, 1)$ and $q^0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Example 4. Another numerical example is

1	−1	2	−3
−1	1	0	1

Here the fourth column dominates the second, and the first column dominates the third. The game is then reduced to

1	−3
−1	1

,

whose value is $-\frac{1}{3}$, and optimal strategies are $p^0 = (\frac{1}{3}, \frac{2}{3})$ and $q^0 = \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \end{pmatrix}$; the latter strategy extends to the strategy

$$\begin{pmatrix} \frac{2}{3} \\ 0 \\ 0 \\ \frac{1}{3} \end{pmatrix},$$

which is optimal in the original game.

Example 5. Our final example shows that there may be a multiplicity of subgames that can be chosen to give optimal strategies. Consider the 4×2 game whose matrix is

13	−7
3	8
−1	14
9	−1

Since there are four rows, there are $\binom{4}{2} = 6$ ways that R can choose a 2×2 subgame. Of these six ways, the one that chooses the first and last row has value -1, and the one that chooses the second and third row has value 3. Each of the other four subgames has value 5. They give rise to the following four optimal strategies for R.

$$(\tfrac{1}{5}, \tfrac{4}{5}, 0, 0)$$
$$(\tfrac{3}{7}, 0, \tfrac{4}{7}, 0)$$
$$(0, 0, \tfrac{2}{5}, \tfrac{3}{5})$$
$$(0, \tfrac{2}{3}, 0, \tfrac{1}{3}).$$

Player C has a unique optimal strategy, namely, $\begin{pmatrix} \frac{3}{5} \\ \frac{2}{5} \end{pmatrix}$.

EXERCISES

1. Solve the following games.

(a)

3	0
−2	3
7	5

[*Ans.* $v = 5; (0, 0, 1); \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.]

(b)

10	5	4	6
18	3	3	4

(c)

1	0	2
0	3	2

[*Ans.* $v = \frac{3}{4}; (\frac{3}{4}, \frac{1}{4}); \begin{pmatrix} \frac{3}{4} \\ \frac{1}{4} \\ 0 \end{pmatrix}$.]

(d)

0	2
1	3
−1	0
2	0

(e)

1	2	3
4	2	1

[*An ans.* $v = 2; (\frac{3}{5}, \frac{2}{5}); \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.]

(f)

1	0	1	1	2
0	−1	−2	−3	−10

2. Solve the following games.

(a)

0	15
8	0
−10	20
10	12

(b)

−1	−2	0	−3	−4
−2	1	0	2	5

(c)

−1	5	−1	−2	8	10
3	−6	0	8	−9	−8

[*An ans.* $v = -\frac{1}{2}$; $(\frac{1}{2}, \frac{1}{2})$; $\begin{pmatrix} 0 \\ \frac{1}{12} \\ \frac{11}{12} \\ 0 \\ 0 \\ 0 \end{pmatrix}$.]

3. Solve the game

1	2	3
3	2	1

.

Since there is more than one optimal strategy for C, find a range of optimal strategies for him. (See Section 7, Exercise 3.)

4. In the card game of Example 2 suppose that R has bk 9, bk 5, rd 7 and rd 3, while C has bk 8 and rd 4. Set up and solve the corresponding matrix game.

[*Ans.* $v = 1$; R shows bk 5 and rd 7 each with probability $\frac{1}{2}$; C shows each of his cards with probability $\frac{1}{2}$.]

5. Suppose that Jones conceals in his hand one, two, three, or four silver dollars and Smith guesses "even" or "odd." If Smith's guess is correct, he wins the amount which Jones holds, otherwise he must pay Jones this amount.

Set up the corresponding matrix game and find an optimal strategy for each player in which he puts positive weight on all his (pure) strategies. Is the game fair?

6. Consider the following game: Player R announces "one" or "two"; then, independently of each other, both players write down one of these two numbers. If the sum of the three numbers so obtained is odd, C pays R the odd sum in dollars; if the sum of the three numbers is even, R pays C the even sum in dollars.

(a) What are the strategies of R? [*Hint:* He has four strategies.]
(b) What are the strategies of C? [*Hint:* We must consider what C does after "one" is announced after a "two." Hence he has four strategies.]
(c) Write the matrix for the game.
(d) Restrict player R to announcing "two," and allow for C only those strategies where his number does not depend on the announced number. Solve the resulting 2×2 game.
(e) Extend the above mixed strategies to the original game, and show that they are optimal.
(f) Is the game favorable to R? If so, by how much?

7. Answer the same questions as in Exercise 6, if R gets the even sum and C gets the odd sum (except that in part (d) restrict R to announce "one"). Which game is more favorable for R? Could you have predicted this without the use of game theory?

8. Rework the five-finger Morra game of Section 6, Exercise 5, with the following payoffs: If the sum of the number of fingers is even, R gets one, while if the sum is odd, C gets one. Suppose that each player shows only one or two fingers. Show that the resulting game is like matching pennies. Show that the optimal strategies for this game, when extended, are optimal in the whole game.

9. A version of three-finger Morra is played as follows: Each player shows from one to three fingers; R always pays C an amount equal to the number of fingers that C shows; if C shows exactly one more or two fewer fingers than R, then C pays R a positive amount x (where x is independent of the number of fingers shown).

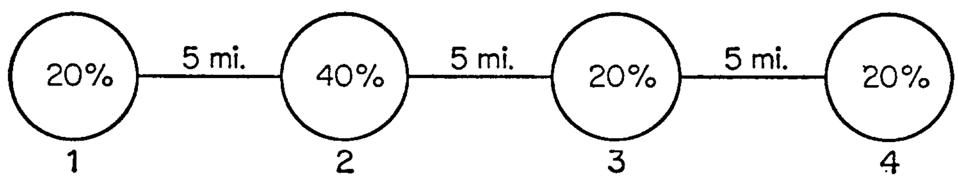
(a) Set up the game matrix for arbitrary x's.
(b) If $x = \frac{1}{2}$, show that the game is strictly determined. Find the value. [*Ans.* $v = -\frac{5}{2}$.]
(c) If $x = 2$, show that there is a pair of optimal strategies in which the first player shows one or two fingers and the second player shows two or three fingers. [*Hint:* Solve a 2×2 derived game.] Find the value. [*Ans.* $v = -\frac{3}{2}$.]

(d) If $x = 6$, show that an optimal strategy for R is to use the mixed strategy $(\frac{1}{3}, \frac{1}{2}, \frac{1}{6})$. Show that the optimal mixed strategy for C is to choose his three strategies each with probability $\frac{1}{3}$. Find the value of the game.

10. Another version of three-finger Morra goes as follows: Each player shows from one to three fingers; if the sum of the number of fingers is even, then R gets an amount equal to the number of fingers that C shows; if the sum is odd, C gets an amount equal to the number of fingers that R shows.

(a) Set up the game matrix.
(b) Reduce the game to a 2×2 matrix game.
(c) Find optimal strategies for each player and show that the game is fair.

11. Two companies, one large and one small, manufacturing the same product, wish to build a new store in one of four towns located on a given highway. If we regard the total population of the four towns as 100 per cent, the distribution of population and distances between towns are as shown.



Assume that if the large company's store is nearer a town it will capture 80 per cent of the business; if both stores are equally distant, then the large company will capture 60 per cent of the business; if the small store is nearer, then the large company will capture 40 per cent of the business.

(a) Set up the matrix of the game.
(b) Test for dominated rows and columns.
(c) Find optimal strategies and value for the game and interpret your results.

[*Ans.* Both companies should locate in town 2; the large company captures 60 per cent of the business.]

12. Rework Exercise 11 if the per cent of business captured by the large company is 90, 75, and 60, respectively.

13. We have stated without proof that any $2 \times n$ game can be solved by considering only its 2×2 derived games. Verify that this is the case for a game of the form $(a > 0, b > 0)$:

C

R			
	a	0	1
	0	b	1

(a) Show that if $a \leq 1$ or $b \leq 1$, then column 3 is dominated. Hence solve the game.

(b) If $a > 1$ and $b > 1$, solve the three 2×2 derived games. [*Hint:* Two of them are strictly determined.]

(c) If $a > 1$, $b > 1$, but $ab < a + b$, then show that the strategies of the nonstrictly determined derived game are optimal for both players.

(d) If $ab \geq a + b$, then show that R has as optimal strategy the same strategy as in part (c), but C has a pure strategy as optimal strategy.

(e) Using the previous results, show that the value of the game is always the smallest of the values of the three derived games.

SUPPLEMENTARY EXERCISES

Exercises 14–20 refer to the *exponential payoff game.* To play the game a number $b > 0$ is chosen. Two sets, S and T, are specified, each set containing at least one positive and at least one negative number (but no zeros). Player R selects a number s from set S, and player C selects a number t from T. If $st > 0$, the payoff is b^{s+t} and if $st < 0$, the payoff is $-b^{s+t}$.

14. Set up the game for $b = 2$ and the sets $S = \{1, -1, 2\}$ and $T = \{1, -3, 2, -4\}$.

[*Ans.*

	1	−3	2	−4
1	4	$-\frac{1}{4}$	8	$-\frac{1}{8}$
−1	−1	$\frac{1}{16}$	−2	$\frac{1}{32}$
2	8	$-\frac{1}{2}$	16	$-\frac{1}{4}$

.]

15. Consider the following mixed strategy for either player: "Choose a positive number p and a negative number n with probabilities $b^n/(b^p + b^n)$ and $b^p/(b^p + b^n)$, respectively." Assume that R uses this strategy.

(a) If C chooses a positive number, show that the expected payoff to R is 0.

(b) If C chooses a negative number, show that the expected payoff to R is 0.

16. Rework Exercise 15 with R and C interchanged.

17. Use the results of Exercises 15 and 16 to show that the game is fair, and that the strategy quoted in Exercise 15 is optimal for either player.

18. Find all strategies of the kind indicated in Exercise 15 for both players for the game of Exercise 14.

[*Partial Ans.* For R they are $(\frac{1}{5}, \frac{4}{5}, 0)$ and $(0, \frac{8}{9}, \frac{1}{9})$.]

19. If a player in the exponential payoff game has m positive and n negative numbers in his set, show that he has mn strategies like those in Exercise 15.

20. Find an optimal strategy for player R which is not of the kind indicated in Exercise 15.

21. Consider the product payoff game described in Exercises 14–20 of Section 7.

(a) If either player has exactly two numbers in his set, show that his optimal strategy is unique.

(b) If either player has more than two numbers in his set, show that he has more than one optimal strategy.

22. Consider the product payoff game described in Exercises 14–20 of Section 7. If player R has two numbers in his set, and C has n (> 2) in his set, show that no column in the resulting game dominates any other column. Do the same for row dominance in the case that C has exactly two numbers in his set.

23. Rework Exercises 21 and 22 for the exponential payoff game.

24. Show that, except for the addition of 5 to each matrix entry, Example 5 is the product payoff game with R choosing from the set $\{4, -1, -3, 2\}$ and C choosing from the set $\{2, -3\}$.

9. *SIMPLIFIED POKER*

In order to illustrate the procedure of translating a game specified by rules into a matrix game, we shall carry it out for a simplification of a well-known game. The example that we are about to discuss is a simplification (by A. W. Tucker) of the poker game discussed on pp. 211–219 in the book *The Theory of Games and Economic Behavior*, by John von Neumann and Oskar Morgenstern.

The deck that is used in simplified poker has only two types of cards, in equal numbers, which we shall call "high" and "low." For example, an ordinary bridge deck could be used with red cards high and black cards low. Each player "antes" an amount a of money and is dealt a single card which is his "hand." By a "deal" we shall mean a pair of cards, the first being given to player R and the second to player C. Thus the deal (H, H) means that each player obtains a high card. There are then four possible deals, namely,

$$(H, H), \quad (H, L), \quad (L, H), \quad (L, L).$$

Ignoring minor errors (see Exercise 1), if the number of cards in the deck is large, each of these deals is "equally likely," that is, the probability of getting a specific one of these deals is $\frac{1}{4}$.

After the deal, player R has the first move and has two alternatives, namely, to "see," or to "raise" by adding an amount b to the pot. If R elects to see, the higher hand wins the pot or equal hands split the pot equally. If R elects to raise, then C has two alternatives, to "fold," or to "call" by adding the amount b to the pot. If C folds, player R wins the pot (without revealing his hand). If C calls, then the higher hand wins the pot or equal hands split the pot equally. These are all the rules.

A pure strategy for a player is a command that tells him exactly what to do in every conceivable situation that can arise in the game. An example of a pure strategy for R is the following: "Raise if you get a high card, and see if you get a low card." We can abbreviate this strategy to simply raise-see. It is easy to see that R has four pure strategies, namely, raise-raise, raise-see, see-raise, and see-see. In the same manner, C has four pure strategies, fold-fold, fold-call, call-fold, call-call.

Given a choice of a pure strategy for each player, there are exactly four ways the play of the game can proceed, depending on which of the four deals occurs. For example, suppose that R has chosen the see-raise strategy, and C has chosen the fold-fold strategy. If the deal is (H, H), then R sees, and they split the pot, so neither wins; if the deal is (H, L), then R sees and wins the pot, giving him a; if the deal is (L, H), then R raises and C folds, so that R wins a; and if the deal is (L, L), then R raises and C folds, so that R wins a. Since the probabilities of each of these deals is $\frac{1}{4}$, the expected value of R's gain is $3a/4$. Let us compute another expected value, namely, suppose that R uses see-raise and C uses call-fold. Then, if the deal is (H, H), R sees and wins nothing; if the deal is (H, L), then R sees and wins a; if the deal is (L, H), then R raises, C calls, and C wins $a + b$; and if the deal is (L, L), then R raises, C folds, and R wins a. The expected value for R here is $(a - b)/4$.

Continuing in this manner, we can compute the expected outcome for each of the 16 possible choices of pairs of strategies. The payoff matrix so obtained is given below.

High		fold	fold	call	call
	Low	fold	call	fold	call
see	see	0	0	0	0
see	raise	$\frac{3a}{4}$	$\frac{2a}{4}$	$\frac{a-b}{4}$	$\frac{-b}{4}$
raise	see	$\frac{a}{4}$	$\frac{a+b}{4}$	0	$\frac{b}{4}$
raise	raise	$\frac{4a}{4}$	$\frac{3a+b}{4}$	$\frac{a-b}{4}$	0

The reader should observe that we have just completed the translation of a game specified by rules into a matrix game.

Since a and b are positive numbers, we see that, in the matrix above, the fourth row dominates the second, and the third row dominates the first. Similarly, the third column dominates the first and second columns. We can reduce the 4×4 matrix to the following 2×2 matrix.

			Conservative	Bluffing
	High		call	call
		Low	fold	call
Conservative	raise	see	0	$\frac{b}{4}$
Bluffing	raise	raise	$\frac{a-b}{4}$	0

Notice that we have labeled the raise-see strategy as "conservative" for R, since it seems sensible to raise when he has a high card and to see when he has a low one. The strategy raise-raise which says, raise even if you have a low card, we have labeled "bluffing," since it corresponds to the ordinary notion of bluffing. In the same manner we have labeled the call-fold strategy "conservative," and the call-call strategy "bluffing," for player C.

Example 1. Suppose $a = 4$ and $b = 8$. Then the matrix becomes

	Conservative	Bluffing
Conservative	0	2
Bluffing	-1	0

Here the game is strictly determined and fair, and optimal strategies are for each player to play conservatively.

Example 2. Suppose $a = 8$ and $b = 4$. Then the matrix becomes

	Conservative	Bluffing
Conservative	0	1
Bluffing	1	0

Here the value of the game is $\frac{1}{2}$, meaning that it is biased in favor of R. Optimal strategies are for each player to bluff with probability $\frac{1}{2}$ and to play conservatively with probability $\frac{1}{2}$.

Here we have one of the most interesting results of game theory, since it turns out that, as part of an optimal strategy, one *should* actually bluff part of the time.

EXERCISES

1. Suppose that the simplified poker game is played with an ordinary bridge deck where red is "high" and black is "low." Compute to four decimal places the conditional probability of drawing a red card, given that one red card has already been drawn. From this, discuss the accuracy of the assumption that the four deals are equally likely. How could the accuracy of the assumption be improved?

2. Substitute $a = 4$ and $b = 8$ into the 4×4 matrix above, and reduce it by dominations to a 2×2 matrix game. Is it the one considered in Example 1 above? Do the same for $a = 8$ and $b = 4$ and compare with Example 2.

3. If $a \leq b$, show that the simplified poker game is strictly determined and fair. Show that both players' optimal strategy is to play conservatively.

4. If $a > b$, show that the simplified poker game is biased in favor of R. Show that, to play optimally, each player must bluff with positive probability, and find the optimal strategies.

5. If $a > b$, discuss ways of making the game fair.

6. When $b \geq a$, show that the optimal strategy of player R is not unique. Show that although he has two "optimal" strategies, the raise-see strategy is in a sense better than the other.

7. Show that in the case $a = 8, b = 4$, the strategy of R can be interpreted as follows: "On a high card always raise, on a low card raise with probability $\frac{1}{2}$." Reinterpret C's mixed strategy similarly.

The remaining exercises concern a variant of the simplified poker game. Real poker is characterized by the fact that there are very many poor hands, and very few good ones. We can make the above model of poker more realistic by making the draw of a low card more probable than that of a high card. Let us say that the probability of drawing a high card is only $\frac{1}{5}$. The rules of the game remain as in the text.

8. Calculate the probabilities of (H, H), (H, L), (L, H), and (L, L) deals.

9. The strategies of the two players are as in the text, hence we will get a similar 4×4 game matrix. Calculate the see-raise *vs.* fold-fold entry of the matrix, just as in the text, but using the results of Exercise 8. Do the same for the see-raise *vs.* call-fold entry. [*Ans.* $24a/25$; $(16a - 4b)/25$.]

10. Fill in the remaining matrix entries.

11. Show that two rows are dominated, and that two columns are dominated.

12. Show that the resulting 2×2 game is strictly determined if and only if $b \geq 4a$. What is the value of the game in these cases?

13. Let $a = 4$, $b = 8$, as in the text, and solve the game. Compare your solution with that in the text.

[*Ans.* Each player should bluff half the time; $v = \frac{16}{25}$; in the previous version there was no bluffing in this case, and the game was fair.]

14. Let $a = 8$, $b = 4$, as in the text, and solve the game. Compare your solution with that in the text.

[*Ans.* Each player plays more conservatively; game is slightly more favorable to R than in the previous version.]

15. The players have agreed that the ante will be $4. They are debating the size of the raise. What value of b should player R argue for? [*Hint:* He does not want the game to be fair. Then what are the possible values of b? Find the value of the 2×2 game for any such b, and find its maximum value by trying several values of b.]

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