Partitions and counting

1. PARTITIONS

The problems to be studied in this chapter can be most conveniently described in terms of partitions of a set. A *partition* of a set $\mathcal{U}$ is a subdivision of the set into subsets that are disjoint and exhaustive, i.e., every element of $\mathcal{U}$ must belong to one and only one of the subsets. The subsets A_i in the partition are called *cells*. Thus $[A_1, A_2, \ldots, A_r]$ is a partition of $\mathcal{U}$ if two conditions are satisfied: (1) $A_i \cap A_j = \mathcal{E}$ if $i \neq j$ (the cells are disjoint) and (2) $A_1 \cup A_2 \cup \ldots \cup A_r = \mathcal{U}$ (the cells are exhaustive).

Example 1. If $\mathcal{U} = \{a, b, c, d, e\}$, then $[\{a, b\}, \{c, d, e\}]$ and $[\{b, c, e\}, \{a\}, \{d\}]$ and $[\{a\}, \{b\}, \{c\}, \{d\}, \{e\}]$ are three different partitions of $\mathcal{U}$. The last is a partition into unit sets.

The process of going from a fine to a less fine analysis of a set of logical possibilities is actually carried out by means of a partition. For example, let us consider the logical possibilities for the first three games of the World Series if the Yankees play the Dodgers. We can list the possibilities in terms of the winner of each game as

{YYY, YYD, YDY, DYY, YDD, DDY, DYD, DDD}.

We form a partition by putting all the possibilities with the same number of wins for the Yankees in a single cell,

$$[\{YYY\}, \{YYD, YDY, DYY\}, \{YDD, DDY, DYD\}, \{DDD\}].$$

Thus, if we wish the possibilities to be Yankees win three games, win two, win one, win zero, then we are considering a less detailed analysis obtained from the former analysis by identifying the possibilities in each cell of the partition.

If $[A_1, A_2, \ldots, A_r]$ and $[B_1, B_2, B_3, \ldots, B_s]$ are two partitions of the same set $\mathcal{U}$, we can obtain a new partition by considering the collection of all subsets of $\mathcal{U}$ of the form $A_i \cap B_j$ (see Exercise 7). This new partition is called the *cross-partition* of the original two partitions.

Example 2. A common use of cross-partitions is in the problem of classification. For example, from the set $\mathcal{U}$ of all life forms we can form the partition $[P, A]$ where P is the set of all plants and A is the set of all animals. We may also form the partition $[E, F]$ where E is the set of extinct life forms and F is the set of all existing life forms. The cross-partition

$$[P \cap E,\quad P \cap F,\quad A \cap E,\quad A \cap F]$$

gives a complete classification according to the two separate classifications.

Many of the examples with which we shall deal in the future will relate to processes which take place in stages. It will be convenient to use partitions and cross-partitions to represent the stages of the process. The graphical representation of such a process is, of course, a tree. For example, suppose that the process is such that we learn in succession the truth values of a series of statements relative to a given situation. If $\mathcal{U}$ is the set of logical possibilities for the situation, and p is a statement relative to $\mathcal{U}$, then the knowledge of the truth value of p amounts to knowing which cell of the partition $[P, \tilde{P}]$ contains the actual possibility. Recall that P is the truth set of p, and $\tilde{P}$ is the truth set of $\sim p$. Suppose now we discover the truth value of a second statement q. This information can again be described by a partition, namely, $[Q, \tilde{Q}]$. The two statements together give us information which can be represented by the cross-partition of these two partitions,

$$[P \cap Q,\quad P \cap \tilde{Q},\quad \tilde{P} \cap Q,\quad \tilde{P} \cap \tilde{Q}].$$

That is, if we know the truth values of p and q, we also know which of the cells of this cross-partition contains the particular logical possibility describing the given situation. Conversely, if we knew which cell contained the possibility, we would know the truth values for the statements p and q.

The information obtained by the additional knowledge of the truth value of a third statement r, having a truth set R, can be represented by the cross-partition of the three partitions $[P, \tilde{P}]$, $[Q, \tilde{Q}]$, $[R, \tilde{R}]$. This cross-partition is

$$[P \cap Q \cap R,\ P \cap Q \cap \tilde{R},\ P \cap \tilde{Q} \cap R,\ \tilde{P} \cap Q \cap R,$$
$$P \cap \tilde{Q} \cap \tilde{R},\ \tilde{P} \cap Q \cap \tilde{R},\ \tilde{P} \cap \tilde{Q} \cap R,\ \tilde{P} \cap \tilde{Q} \cap \tilde{R}].$$

Notice that now we have the possibility narrowed down to being in one of $8 = 2^3$ possible cells. Similarly, if we knew the truth values of n statements, our partition would have 2^n cells.

If the set $\mathcal{U}$ were to contain 2^{20} (approximately one million) logical possibilities, and if we were able to ask yes-no questions in such a way that the knowledge of the truth value of each question would cut the number of possibilities in half each time, then we could determine in 20 questions any given possibility in the set $\mathcal{U}$. We could accomplish this kind of questioning, for example, if we had a list of all the possibilities and were allowed to ask "Is it in the first half?" and, if the answer is yes, then "Is it in the first one-fourth?," etc. In practice we ordinarily do not have such a list, and we can only approximate this procedure.

Example 3. In the familiar radio game of twenty questions it is not unusual for a contestant to try to carry out a partitioning of the above kind. For example, he may know that he is trying to guess a city. He might ask, "Is the city in North America?" and if the answer is yes, "Is it in the United States?" and if yes, "Is it west of the Mississippi?" and if no, "Is it in the New England states?," etc. Of course, the above procedure does not actually divide the possibilities exactly in half each time. The more nearly the answer to each question comes to dividing the possibilities in half, the more certain one can be of getting the answer in twenty questions, if there are at most a million possibilities.

EXERCISES

1. If $\mathcal{U}$ is the set of integers from 1 to 6, find the cross-partitions of the following pairs of partitions.

(a) $[\{1, 2, 3\}, \{4, 5, 6\}]$ and $[\{1, 4\}, \{2, 3, 5, 6\}]$.

(b) $[\{1, 2, 3, 4, 5\}, \{6\}]$ and $[\{1, 3, 5\}, \{2, 6\}, \{4\}]$.

[*Ans.* (a) $\{1\}, \{2, 3\}, \{4\}, \{5, 6\}$.]

2. A coin is thrown three times. List the possibilities according to which side turns up each time. Give the partition formed by putting in the same cell all those possibilities for which the same number of heads occur.

3. Let p and q be two statements with truth set P and Q. What can be said about the cross-partition of $[P, \tilde{P}]$ and $[Q, \tilde{Q}]$ in the case that

(a) p implies q. [*Ans.* $P \cap \tilde{Q} = \varepsilon$.]

(b) p is equivalent to q.

(c) p and q are inconsistent.

4. Consider the set of eight states consisting of Illinois, Colorado, Michigan, New York, Vermont, Texas, Alabama, and California.

(a) Show that in three "yes" or "no" questions one can identify any one of the eight states.

(b) Design a set of three "yes" or "no" questions which can be answered independently of each other and which will serve to identify any one of the states.

5. An unabridged dictionary contains about 600,000 words and 3000 pages. If a person chooses a word from such a dictionary, is it possible to identify this word by twenty "yes" or "no" questions? If so, describe the procedure that you would use and discuss the feasibility of the procedure.

[*Ans.* One solution is the following. Use 12 questions to locate the page, but then you may need 9 questions to locate the word.]

6. Mr. Jones has two parents, each of his parents had two parents, each of these had two parents, etc. Tracing a person's family tree back 40 generations (about 1000 years) gives Mr. Jones 2^{40} ancestors, which is more people than have been on the earth in the last 1000 years. What is wrong with this argument?

7. Let $[A_1, A_2, A_3]$ and $[B_1, B_2]$ be two partitions. Prove that the cross-partition of the two given partitions really is a partition, that is, it satisfies requirements (1) and (2) for partitions.

8. The cross-partition formed from the truth sets of n statements has 2^n cells. As seen in Chapter I, the truth table of a statement compounded from n statements has 2^n rows. What is the relationship between these two facts?

9. Let p and q be statements with truth sets P and Q. Form the partition $[P \cap Q, P \cap \tilde{Q}, \tilde{P} \cap Q, \tilde{P} \cap \tilde{Q}]$. State in each case below which of the cells must be empty in order to make the given statement a logically true statement.

(a) $p \rightarrow q$.
(b) $p \leftrightarrow q$.
(c) $p \vee \sim p$.
(d) p.

10. A partition $[A_1, A_2, \ldots, A_n]$ is said to be a *refinement* of the partition $[B_1, B_2, \ldots, B_m]$ if every A_j is a subset of some B_k. Show that a cross-partition of two partitions is a refinement of each of the partitions from which the cross-partition is formed.

11. Consider the partition of the people in the United States determined by classification according to states. The classification according to county determines a second partition. Show that this is a refinement of the first partition. Give a third partition which is different from each of these and is a refinement of both.

12. What can be said concerning the cross-partition of two partitions, one of which is a refinement of the other?

13. Given nine objects, of which it is known that eight have the same weight and one is heavier, show how, in two weighings with a pan balance, the heavy one can be identified.

14. Suppose that you are given thirteen objects, twelve of which are the same, but one is either heavier or lighter than the others. Show that, with three weighings using a pan balance, it is possible to identify the odd object. [A complete solution to this problem is given on page 42 of *Mathematical Snapshots*, second edition, by H. Steinhaus.]

15. A subject can be completely classified by introducing several simple subdivisions and taking their cross-partition. Thus, courses in college may be partitioned according to subject, level of advancement, number of students, hours per week, interests, etc. For each of the following subjects, introduce five or more partitions. How many cells are there in the complete classification (cross-partition) in each case?

(a) Detective stories.
(b) Diseases.

16. Assume that in a given generation x men are Republicans and y are Democrats and that the total number of men remains at 50 million in each generation. Assume further that it is known that 20 per cent of the sons of Republicans are Democrats and 30 per cent of the sons of Democrats are Republicans in any generation. What conditions must x and y satisfy if there are to be the same number of Republicans in each generation? Is there more than one choice for x and y? If not, what must x and y be?

[*Partial Ans.* There are 30 million Republicans.]

17. Assume that there are 30 million Democratic and 20 million Republican men in the country. It is known that p per cent of the sons of Democrats are Republicans, and q per cent of the sons of Republicans are Democrats. If the total number of men remains 50 million, what condition must p and q satisfy so that the number in each party remains the same? Is there more than one choice of p and q?

2. *THE NUMBER OF ELEMENTS IN A SET*

The remainder of this chapter will be devoted to certain counting problems. For any set X we shall denote by $n(X)$ the number of elements in the set.

Suppose we know the number of elements in certain given sets and wish to know the number in other sets related to these by the operations of unions, intersections, and complementations. As an example, consider the following problem.

Suppose that we are told that 100 students take mathematics, and 150 students take economics. Can we then tell how many take either mathematics or economics? The answer is no, since clearly we would also need to know how many students take both courses. If we know that no student takes both courses, i.e., if we know that the two sets of students are disjoint, then the answer would be the sum of the two numbers or 250 students.

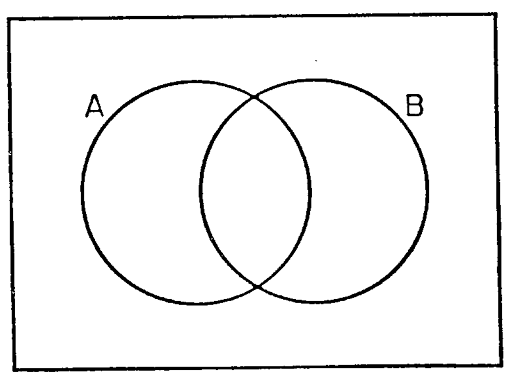

Figure 1

In general, if we are given disjoint sets A and B, then it is true that $n(A \cup B) = n(A) + n(B)$. Suppose now that A and B are not disjoint as shown in Figure 1. We can divide the set A into disjoint sets $A \cap \tilde{B}$ and $A \cap B$. Similarly, we can divide B into the disjoint sets $\tilde{A} \cap B$ and $A \cap B$. Thus,

$$n(A) = n(A \cap \tilde{B}) + n(A \cap B)$$
$$n(B) = n(\tilde{A} \cap B) + n(A \cap B).$$

Adding these two equations, we obtain

$$n(A) + n(B) = n(A \cap \tilde{B}) + n(\tilde{A} \cap B) + 2n(A \cap B).$$

Since the sets $A \cap \tilde{B}$, $\tilde{A} \cap B$, and $A \cap B$ are disjoint sets whose union is $A \cup B$, we obtain the formula

$$n(A \cup B) = n(A) + n(B) - n(A \cap B),$$

which is valid for any two sets A and B.

Example 1. Let p and q be statements relative to a set $\mathcal{U}$ of logical possibilities. Denote by P and Q the truth sets of these statements. The truth set of $p \vee q$ is $P \cup Q$ and the truth set of $p \wedge q$ is $P \cap Q$. Thus the above formula enables us to find the number of cases where $p \vee q$ is true if we know the number of cases for which p, q, and $p \wedge q$ are true.

Example 2. *More than two sets.* It is possible to derive formulas for the number of elements in a set which is the union of more than two sets (see Exercise 6), but usually it is easier to work with Venn diagrams. For example, suppose that the registrar of a school reports the following statistics about a group of 30 students:

19 take mathematics.
17 take music.
11 take history.
12 take mathematics and music.
7 take history and mathematics.
5 take music and history.
2 take mathematics, history, and music.

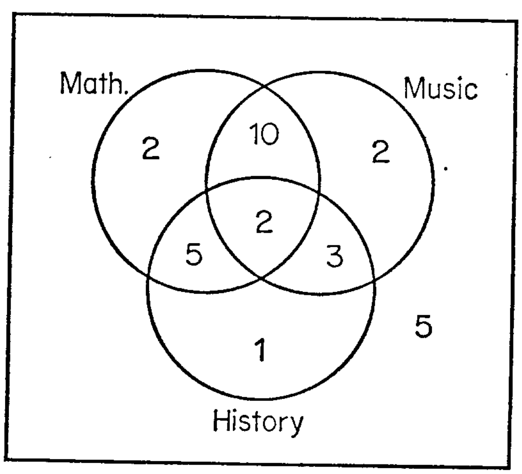

Figure 2

We draw the Venn diagram in Figure 2 and fill in the numbers for the number of elements in each subset working from the bottom of our list to the top. That is, since 2 students take all three courses, and 5 take music and history, then 3 take history and music but not mathematics, etc. Once the diagram is completed we can read off the number which take any combination of the courses. For example, the number which take history but not mathematics is $3 + 1 = 4$.

Example 3. *Cancer studies.* The following reasoning is often found in statistical studies on the effect of smoking on the incidence of lung cancer. Suppose a study has shown that the fraction of smokers among those who have lung cancer is greater than the fraction of smokers among those who do not have lung cancer. It is then asserted that the fraction of smokers who have lung cancer is greater than the fraction of nonsmokers who have lung cancer. Let us examine this argument.

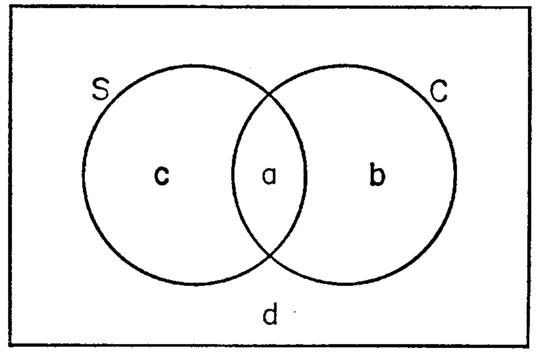

Figure 3

Let S be the set of all smokers in the population, and C be the set of all people with lung cancer. Let $a = n(S \cap C)$, $b = n(\tilde{S} \cap C)$, $c = n(S \cap \tilde{C})$, and $d = n(\tilde{S} \cap \tilde{C})$, as indicated in Figure 3. The fractions in which we are interested are

$$p_1 = \frac{a}{a+b}, \quad p_2 = \frac{c}{c+d}, \quad p_3 = \frac{a}{a+c}, \quad p_4 = \frac{b}{b+d},$$

where p_1 is the fraction of those with lung cancer that smoke, p_2 the fraction of those without lung cancer that smoke, p_3 the fraction of smokers who have lung cancer, and p_4 the fraction of nonsmokers who have cancer.

The argument above states that if $p_1 > p_2$, then $p_3 > p_4$. The hypothesis,

$$\frac{a}{a+b} > \frac{c}{c+d}$$

is true if and only if $ac + ad > ac + bc$, that is, if and only if $ad > bc$. The conclusion

$$\frac{a}{a+c} > \frac{b}{b+d}$$

is true if and only if $ab + ad > ab + bc$, that is, if and only if $ad > bc$. Thus the two statements $p_1 > p_2$ and $p_3 > p_4$ are in fact equivalent statements, so that the argument is valid.

EXERCISES

1. In Example 2, find

(a) The number of students that take mathematics but do not take history. [*Ans.* 12.]

(b) The number that take exactly two of the three courses.

(c) The number that take one or none of the courses.

2. In a chemistry class there are 20 students, and in a psychology class there are 30 students. Find the number in either the psychology class or the chemistry class if

(a) The two classes meet at the same hour. [*Ans.* 50.]

(b) The two classes meet at different hours and 10 students are enrolled in both courses. [*Ans.* 40.]

3. If the truth set of a statement p has 10 elements, and the truth set of a statement q has 20 elements, find the number of elements in the truth set of $p \vee q$ if

(a) p and q are inconsistent.

(b) p and q are consistent and there are two elements in the truth set of $p \wedge q$.

4. If p is a statement that is true in ten cases, and q is a statement that is true in five cases, find the number of cases in which both p and q are true if $p \vee q$ is true in ten cases. What relation holds between p and q?

5. Assume that the incidence of lung cancer is 16 per 100,000, and that it is estimated that 75 per cent of those with lung cancer smoke and 60 per cent of those without lung cancer smoke. (These numbers are fictitious.) Estimate the fraction of smokers with lung cancer, and the fraction of nonsmokers with lung cancer. [*Ans.* 20 and 10 per 100,000.]

6. Let A, B, and C be any three sets of a universal set $\mathcal{U}$. Draw a Venn diagram and show that

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C).$$

7. Analyze the data given below and draw a Venn diagram like that in Figure 2. Assuming that every student in the school takes one of the courses, find the total number of students in the school.

(a)	(b)	
28	36	students take English.
23	23	students take French.

23	13	students take German.
12	6	students take English and French.
11	11	students take English and German.
8	4	students take French and German.
5	1	students take all three courses.

Comment on the result in (b).

8. Suppose that in a survey concerning the reading habits of students it is found that:

60 per cent read magazine A.
50 per cent read magazine B.
50 per cent read magazine C.
30 per cent read magazines A and B.
20 per cent read magazines B and C.
30 per cent read magazines A and C.
10 per cent read all three magazines.

(a) What per cent read exactly two magazines? [*Ans.* 50.]
(b) What per cent do not read any of the magazines? [*Ans.* 10.]

9. If p and q are equivalent statements and $n(P) = 10$, what is $n(P \cup Q)$?

10. If p implies q, prove that $n(P \cup \tilde{Q}) = n(P) + n(\tilde{Q})$.

11. On a transcontinental airliner, there are 9 boys, 5 American children, 9 men, 7 foreign boys, 14 Americans, 6 American males, and 7 foreign females. What is the number of people on the plane? [*Ans.* 33.]

SUPPLEMENTARY EXERCISES

12. Prove that $n(\tilde{A}) = n(\mathcal{U}) - n(A)$.

13. Show that $n(\tilde{A} \cap \tilde{B}) = n(\widetilde{A \cup B}) = n(\mathcal{U}) - n(A \cup B)$.

14. In a collection of baseball players there are ten who can play only outfield positions, five who can play only infield positions but cannot pitch, three who can pitch, four who can play any position but pitcher, and two who can play any position at all. How many players are there in all? [*Ans.* 22.]

15. Ivyten College awarded 38 varsity letters in football, 15 in basketball, and 20 in baseball. If these letters went to a total of 58 men and only three of these men lettered in all three sports, how many men received letters in exactly two of the three sports? [*Ans.* 9.]

16. Let $\mathcal{U}$ be a finite set. For any two sets A and B define the "distance" from A to B to be $d(A, B) = n(A \cap \tilde{B}) + n(\tilde{A} \cap B)$.

(a) Show that $d(A, B) \geq 0$. When is $d(A, B) = 0$?

(b) If A, B, and C are nonintersecting sets, show that

$$d(A, C) \leq d(A, B) + d(B, C).$$

(c) Show that for any three sets A, B, and C

$$d(A, C) \leq d(A, B) + d(B, C).$$

3. *PERMUTATIONS*

We wish to consider here the number of ways in which a group of n different objects can be arranged. A listing of n different objects *in a certain order* is called a *permutation* of the n objects. We consider first the case of three objects, a, b, and c. We can exhibit all possible permutations of these three objects as paths of a tree, as shown in Figure 4. Each path exhibits a possible permutation, and there are six such paths. We could also list these permutations as follows:

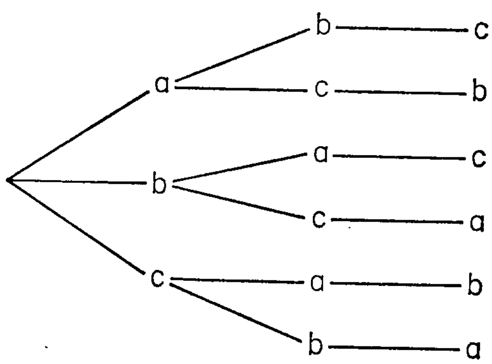

Figure 4

abc,	bca,
acb,	cab,
bac,	cba.

If we were to construct a similar tree for n objects, we would find that the number of paths could be found by multiplying together the numbers n, $n - 1$, $n - 2$, continuing down to the number 1. The number obtained in this way occurs so often that we give it a symbol, namely $n!$, which is read "n factorial." Thus, for example, $3! = 3 \cdot 2 \cdot 1 = 6$, $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$, etc. For reasons which will be clear later, we define $0! = 1$. Thus we can say *there are* $n!$ *different permutations of* n *distinct objects.*

Example 1. In the game of Scrabble, suppose there are seven lettered blocks from which we try to form a seven-letter word. If the seven letters are all different, we must consider $7! = 5040$ different orders.

Example 2. A quarterback has a sequence of ten plays. Suppose his coach instructs him to run through the ten-play sequence without repetition. How much freedom is left to the quarterback? He may choose any one of $10! = 3{,}628{,}800$ orders in which to call the plays.

Example 3. How many ways can n people be seated around a circular table? When this question is asked, it is usually understood that two arrangements are different only if at least one person has a different person next to him in the two arrangements. Consider then one person in a fixed position. There are $(n - 1)!$ ways in which the other people may be seated. We have now counted all the arrangements we wish to consider different.

A general principle. There are many counting problems for which it is not possible to give a simple formula for the number of possible cases. In many of these the only way to find the number of cases is to draw a tree and count them (see Exercise 4). In some problems, the following general principle is useful.

If one thing can be done in exactly r different ways, for each of these a second thing can be done in exactly s different ways, for each of the first two, a third can be done in exactly t ways, etc., then the sequence of things can be done in the product of the numbers of ways in which the individual things can be done, i.e., $r \cdot s \cdot t \ldots$ *ways.*

The validity of the above general principle can be established by thinking of a tree representing all the ways in which the sequence of things can be done. There would be r branches from the starting position. From the ends of each of these r branches there would be s new branches, and from each of these t new branches, etc. The number of paths through the tree would be given by the product $r \cdot s \cdot t \ldots.$

Example 4. The number of permutations of n distinct objects is a special case of this principle. If we were to list all the possible permutations, there would be n possibilities for the first, for each of these $n - 1$ for the second, etc., until we came to the last object, and for which there is only one possibility. Thus there are $n(n - 1) \ldots 1 = n!$ possibilities in all.

Example 5. If there are three roads from city x to city y and two roads from city y to city z, then there are $3 \cdot 2 = 6$ ways that a person can drive from city x to city z passing through city y.

Example 6. Suppose there are n applicants for a certain job. Three interviewers are asked independently to rank the applicants according to their suitability for the job. It is decided that an applicant will be hired if he is ranked first by at least two of the three interviewers. What fraction of the possible reports would lead to the acceptance of some candidate? We shall solve this problem by finding the fraction of the reports which do not lead to an acceptance and subtract this answer from 1. Frequently, an indirect attack of this kind on a problem is easier than the direct approach. The total number of reports possible is $(n!)^3$ since each interviewer can rank the men in $n!$ different ways. If a particular report does not lead to the acceptance of a candidate, it must be true that each interviewer has put a different man in first place. This can be done in $n(n - 1)(n - 2)$ different ways by our general principle. For each possible first choices, there are $[(n - 1)!]^3$ ways in which the remaining men can be ranked by the interviewers. Thus the number of reports which do not lead to acceptance is $n(n - 1)(n - 2)[(n - 1)!]^3$. Dividing this number by $(n!)^3$ we obtain

$$\frac{(n - 1)(n - 2)}{n^2}$$

as the fraction of reports which fail to accept a candidate. The fraction which leads to acceptance is found by subtracting this fraction from 1 which gives

$$\frac{3n - 2}{n^2}.$$

For the case of three applicants, we see that $\frac{7}{9}$ of the possibilities lead to acceptance. Here the procedure might be criticized on the grounds that even if the interviewers are completely ineffective and are essentially guessing there is a good chance that a candidate will be accepted on the basis of the reports. For n equal to ten, the fraction of acceptances is only .28, so that it is possible to attach more significance to the interviewers ratings, if they reach a decision.

EXERCISES

1. In how many ways can five people be lined up in a row for a group picture? In how many ways if it is desired to have three in the front row and two in the back row? [*Ans.* 120; 120.]

2. Assuming that a baseball team is determined by the players and the position each is playing, how many teams can be made from 13 players if
   (a) Each player can play any position?
   (b) Two of the players can be used only as pitchers?

3. Grades of A, B, C, D, or E are assigned to a class of five students.
   (a) How many ways may this be done, if no two students receive the same grade? [*Ans.* 120.]
   (b) Two of the students are named Smith and Jones. How many ways can grades be assigned if no two students receive the same grade and Smith must receive a higher grade than Jones? [*Ans.* 60.]
   (c) How many ways may grades be assigned if only grades of A and E are assigned? [*Ans.* 32.]

4. A certain club wishes to admit seven new members, four of whom are Republicans and three of whom are Democrats. Suppose the club wishes to admit them one at a time and in such a way that there are always more Republicans among the new members than there are Democrats. Draw a tree to represent all possible ways in which new members can be admitted, distinguishing members by their party only.

5. There are three different routes connecting city A to city B. How many ways can a round trip be made from A to B and back? How many ways if it is desired to take a different route on the way back? [*Ans.* 9; 6.]

6. How many different ways can a ten-question multiple-choice exam be answered if each question has three possibilities, a, b, and c? How many if no two consecutive answers are the same?

7. Modify Example 6 so that, to be accepted, an applicant must be first in two of the interviewers' ratings and must be either first or second in the third interviewers' rating. What fraction of the possible reports lead to acceptance in the case of three applicants? In the case of n? [*Ans.* $\frac{4}{9}$; $4/n^2$.]

8. A town has 1240 registered Republicans. It is desired to contact each of these by phone to announce a meeting. A committee of r people devise a method of phoning s people each and asking each of these to call t new people. If the method is such that no person is called twice,
   (a) How many people know about the meeting after the phoning?

(b) If the committee has 40 members and it is desired that all 1240 Republicans be informed of the meeting and that s and t should be the same, what should they be?

9. In the Scrabble example, suppose the letters are Q, Q, U, F, F, F, A. How many distinguishable arrangements are there for these seven letters?

[*Ans.* 420.]

10. How many different necklaces can be made

(a) If seven different sized beads are available? [*Ans.* 360.]

(b) If six of the beads are the same size and one is larger? [*Ans.* 1.]

(c) If the beads are of two sizes, five of the smaller size and two of the larger size? [*Ans.* 3.]

11. Prove that two people in Columbus, Ohio, have the same initials.

12. Find the number of arrangements of the five symbols that can be distinguished. (The same letters with different subscripts indicate distinguishable objects.)

(a) A_1, A_2, B_1, B_2, B_3. [*Ans.* 120.]

(b) A, A, B_1, B_2, B_3. [*Ans.* 60.]

(c) A, A, B, B, B. [*Ans.* 10.]

13. Show that the number of distinguishable arrangements possible for n objects, n_1 of type 1, n_2 of type 2, etc., for r different types is

$$\frac{n!}{n_1!\, n_2! \ldots, n_r!}.$$

SUPPLEMENTARY EXERCISES

14. (a) How many four digit numbers can be formed from the digits 1, 2, 3, 4, using each digit only once?

(b) How many of these numbers are less than 3000? [*Ans.* 12.]

15. How many license plates can be made if they are to contain five symbols, the first two being letters and the last three integers?

16. How many signals can a ship show if it has seven flags and a signal consists of five flags hoisted vertically on a rope? [*Ans.* 2520.]

17. We must arrange three green, two red, and four blue books on a single shelf.

(a) In how many ways can this be done if there are no restrictions?

(b) In how many ways if books of the same color must be grouped together?

(c) In how many ways if, in addition to the restriction in (b), the red books must be to the left of the blue books?

(d) In how many ways if, in addition to the restrictions in (b) and (c), the red and blue books must not be next to each other? [*Ans*. 288.]

18. A young lady has three shades of nail polish with which to paint her fingernails. In how many ways can she do this (each nail being one solid color) if there are no more than two different shades on each hand?

[*Ans*. 8649.]

4. COUNTING PARTITIONS

Up to now we have not had occasion to consider the partitions $[\{1, 2\}, \{3, 4\}]$ and $[\{3, 4\}, \{1, 2\}]$ of the integers from 1 to 4 as being different partitions. Here it will be convenient to do so, and to indicate this distinction we shall use the term *ordered partition.* An *ordered partition with r cells* is a partition with r cells (some of which may be empty), with a particular order specified for the cells.

We are interested in counting the number of possible ordered partitions with r cells that can be formed from a set of n objects having a prescribed number of elements in each cell. We consider first a special case to illustrate the general procedure.

Suppose that we have eight students, A, B, C, D, E, F, G, and H, and we wish to assign these to three rooms, Room 1, which is a triple room, Room 2, a triple room, and Room 3, a double room. In how many different ways can the assignment be made? One way to assign the students is to put them in the rooms in the order in which they arrive, putting the first three in Room 1, the next three in Room 2, and the last two in Room 3. There are 8! ways in which the students can arrive, but not all of these lead to different assignments. We can represent the assignment corresponding to a particular order of arrival as follows,

$$|\text{BCA}|\text{DFE}|\text{HG}|.$$

In this case, B, C, and A are assigned to Room 1, D, F, and E to Room 2, and H and G to Room 3. Notice that orders of arrival which simply change the order within the rooms lead to the same assignment. The number of different orders of arrival which lead to the same assignment as the one above is the number of arrangements which differ from the given one only in that the arrangement within the rooms is different. There are $3!\cdot 3!\cdot 2!$ such orders of arrival, since we can arrange the three in Room 1 in 3! different ways, for each of these the ones in Room 2 in 3! different ways, and for each of these, the ones in Room 3 in 2! ways. Thus we can divide the 8! different orders of arrival into groups of $3!\cdot 3!\cdot 2!$ different orders such that all the orders of arrival in a single group lead to the same room assignment. Since there are $3!\cdot 3!\cdot 2!$ elements in each group and 8! elements altogether, there are $\frac{8!}{3!\,3!\,2!}$ groups, or this many different room assignments.

The same argument could be carried out for n elements and r rooms, with n_1 in the first, n_2 in the second, etc. This would lead to the following result. Let $n_1, n_2, \ldots, n_r$ be nonnegative integers with

$$n_1 + n_2 + \ldots + n_r = n.$$

Then:

The number of ordered partitions with r cells $[A_1, A_2, A_3, \ldots, A_r]$ of a set of n elements with n_1 in the first cell, n_2 in the second, etc. is

$$\frac{n!}{n_1!\, n_2! \ldots n_r!}.$$

We shall denote this number by the symbol

$$\binom{n}{n_1, n_2, \ldots, n_r}.$$

Note that this symbol is defined only if $n_1 + n_2 + \ldots + n_r = n$.

The special case of two cells is particularly important. Here the problem can be stated equivalently as the problem of finding the number of subsets with r elements that can be chosen from a set of n elements. This is true because any choice defines a partition $[A, \tilde{A}]$, where A is the set of elements chosen and $\tilde{A}$ is the set of remaining elements. The number of such partitions is $\frac{n!}{r!(n-r)!}$ and hence this is also the number of subsets with r elements. Our notation $\binom{n}{r, n-r}$ for this case is shortened to $\binom{n}{r}$.

Notice that $\binom{n}{n-r}$ is the number of subsets with $n - r$ elements which can be chosen from n, which is the number of partitions of the form $[\tilde{A}, A]$ above. Clearly, this is the same as the number of $[A, A]$ partitions. Hence $\binom{n}{r} = \binom{n}{n-r}$.

Example 1. A college has scheduled six football games during a season. How many ways can the season end in two wins, three losses, and one tie? From each possible outcome of the season, we form a partition, with three cells, of the opposing teams. In the first cell we put the teams which our college defeats, in the second the teams to which our college loses, and in the third cell the teams which our college ties. There are $\binom{6}{2, 3, 1} = 60$ such partitions, and hence 60 ways in which the season can end with two wins, three losses, and one tie.

Example 2. In the game of bridge, the hands N, E, S, and W determine a partition of the 52 cards having four cells, each with 13 elements. Thus there are $\frac{52!}{13!\,13!\,13!\,13!}$ different bridge deals. This number is about $5.3645 \cdot 10^{28}$ or approximately 54 billion billion billion deals.

Example 3. The following example will be important in probability theory, which we take up in the next chapter. If a coin is thrown six times, there are 2^6 possibilities for the outcome of the six throws, since each throw can result in either a head or a tail. How many of these possibilities result in four heads and two tails? Each sequence of six heads and tails determines a two-cell partition of the numbers from one to six as follows: In the first cell put the numbers corresponding to throws which resulted in a head, and in the second put the numbers corresponding to throws which resulted in tails. We require that the first cell should contain four elements and the second two elements. Hence the number of the 2^6 possibilities which lead to four heads and two tails is the number of two-cell partitions of six elements which have four elements in the first cell and two in the second cell. The answer is $\binom{6}{4} = 15$. For n throws of a coin, a similar analysis shows that there are $\binom{n}{r}$ different sequences of H's and T's of length n which have exactly r heads and $n - r$ tails.

EXERCISES

1. Compute the following numbers.

(a) $\binom{7}{5}$ [*Ans.* 21.]

(b) $\binom{3}{2}$

(c) $\binom{7}{2}$

(d) $\binom{250}{249}$ [*Ans.* 250.]

(e) $\binom{5}{0}$

(f) $\binom{5}{1, 2, 2}$

(g) $\binom{4}{2, 0, 2}$ [*Ans.* 6.]

(h) $\binom{2}{1, 1, 1}$

2. Give an interpretation for $\binom{n}{0}$ and also for $\binom{n}{n}$. Can you now give a reason for making $0! = 1$?

3. How many ways can nine students be assigned to three triple rooms? How many ways if one particular pair of students refuse to room together? [*Ans.* 1680; 1260.]

4. A group of seven boys and ten girls attends a dance. If all the boys dance in a particular dance, how many choices are there for the girls who dance? For the girls who do not dance? How many choices are there for the girls who do not dance, if three of the girls are sure to be asked to dance?

5. Suppose that a course is given at three different hours. If fifteen students sign up for the course,

(a) How many possibilities are there for the ways the students could distribute themselves in the classes? [*Ans.* 3^{15}.]

(b) How many of the ways would give the same number of students in each class? [*Ans.* 756,756.]

6. A college professor anticipates teaching the same course for the next 35 years. So not to become bored with his jokes, he decides to tell exactly three jokes every year and in no two years to tell exactly the same three jokes. What is the minimum number of jokes that will accomplish this? What is the minimum number if he determines never to tell the same joke twice?

7. How many ways can you answer a ten-question true-false exam, marking the same number of answers true as you do false? How many if it is desired to have no two consecutive answers the same?

8. From three Republicans and three Democrats, find the number of committees of three which can be formed

(a) With no restrictions. [*Ans.* 20.]

(b) With three Republicans and no Democrats. [*Ans.* 1.]
(c) With two Republicans and one Democrat. [*Ans.* 9.]
(d) With one Republican and two Democrats. [*Ans.* 9.]
(e) With no Republicans and three Democrats. [*Ans.* 1.]

What is the relation between your answer in part (a) and the answers to the remaining four parts?

9. Exercise 8 suggests that the following should be true.

$$\binom{2n}{n} = \binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \binom{n}{2}\binom{n}{n-2} + \cdots + \binom{n}{n}\binom{n}{0}$$
$$= \binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2.$$

Show that it is true.

10. A student needs to choose two electives from six possible courses.
(a) How many ways can he make his choice? [*Ans.* 15.]
(b) How many ways can he choose if two of the courses meet at the same time? [*Ans.* 14.]
(c) How many ways can he choose if two of the courses meet at 10 o'clock, two at 11 o'clock, and there are no other conflicts among the courses? [*Ans.* 13.]

SUPPLEMENTARY EXERCISES

11. Consider a town in which there are three plumbers, A, B, and C. On a certain day six residents of the town telephone for a plumber. If each resident selects a plumber from the telephone directory, in how many ways can it happen that
(a) Three residents call A, two residents call B, and one resident calls C? [*Ans.* 60.]
(b) The distribution of calls to the plumbers is three, two, and one? [*Ans.* 360.]

12. Two committees (a labor relations committee and a quality control committee) are to be selected from a board of nine men. The only rules are (1) the two committees must have no members in common, and (2) each committee must have at least four men. In how many ways can the two committees be appointed?

13. A group of ten people is to be divided into three committees of three, three, and six members, respectively. The chairman of the group is to serve on all three committees and is the only member of the group who serves on more than one committee. In how many ways can the committee assignments be made? [*Ans.* 756.]

14. In a class of 20 students, grades of A, B, C, D, and F are to be assigned. Omit arithmetic details in answering the following.

(a) In how many ways can this be done if there are no restrictions? [*Ans.* 5^{20}.]

(b) In how many ways can this be done if the grades are assigned as follows: 2 A's, 3 B's, 10 C's, 3 D's, and 2 F's?

(c) In how many ways can this be done if the following rules are to be satisfied: exactly 10 C's; the same number of A's as F's; the same number of B's as D's; always more B's than A's?

$$[\textit{Ans.}\ \binom{20}{5,\,10,\,5} + \binom{20}{1,\,4,\,10,\,4,\,1} + \binom{20}{2,\,3,\,10,\,3,\,2}.]$$

15. Establish the identity

$$\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}$$

for $n \geq r \geq k$ in two ways, as follows:

(a) Replace each expression by a ratio of factorials and show that the two sides are equal.

(b) Consider the following problem: From a set of n people a committee of r is to be chosen, and from these r people a steering subcommittee of k people is to be selected. Show that the two sides of the identity give two different ways of counting the possibilities for this problem.

5. SOME PROPERTIES OF THE NUMBERS $\binom{n}{j}$

The numbers $\binom{n}{j}$ introduced in the previous section will play an important role in our future work. We give here some of the more important properties of these numbers.

A convenient way to obtain these numbers is given by the famous Pascal triangle, shown in Figure 5. To obtain the triangle we first write the 1's down the sides. Any of the other numbers in the triangle has the property that it is the sum of the two adjacent numbers in the row just above. Thus the next row in the triangle is 1, 6, 15, 20, 15, 6, 1. To find the number $\binom{n}{j}$ we look in the row corresponding to the number n and see where the diagonal line corresponding to the

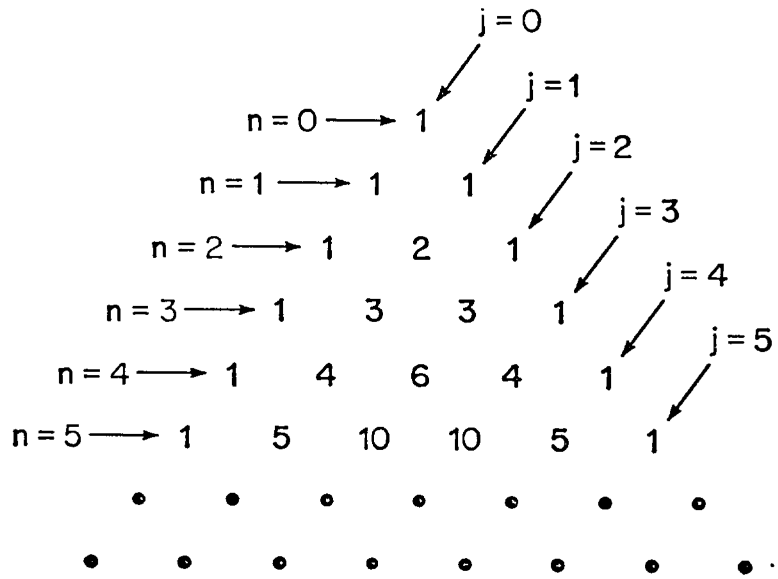

Figure 5

value of j intersects this row. For example, $\binom{4}{2} = 6$ is in the row marked $n = 4$ and on the diagonal marked $j = 2$.

The property of the numbers $\binom{n}{j}$ upon which the triangle is based is

$$\binom{n+1}{j} = \binom{n}{j-1} + \binom{n}{j}.$$

This fact can be verified directly (see Exercise 6), but the following argument is interesting in itself. The number $\binom{n+1}{j}$ is the number of subsets with j elements that can be formed from a set of $n + 1$ elements. Select one of the $n + 1$ elements, x. The $\binom{n+1}{j}$ subsets can be partitioned into those that contain x and those that do not. The latter are subsets of j elements formed from n objects, and hence there are $\binom{n}{j}$ such subsets. The former are constructed by adding x to a subset of $j - 1$ elements formed from n elements, and hence there are $\binom{n}{j-1}$ of them. Thus

$$\binom{n+1}{j} = \binom{n}{j-1} + \binom{n}{j}.$$

If we look again at the Pascal triangle, we observe that the numbers in a given row increase for a while, and then decrease. We can prove this fact in general by considering the ratio of two successive terms,

$$\frac{\binom{n}{j+1}}{\binom{n}{j}} = \frac{n!}{(j+1)!\,(n-j-1)!} \cdot \frac{j!\,(n-j)!}{n!} = \frac{n-j}{j+1}.$$

The numbers increase as long as the ratio is greater than 1, i.e., $n - j > j + 1$. This means that $j < \frac{1}{2}(n-1)$. We must distinguish the case of an even n from an odd n. For example, if $n = 10$, j must be less than $\frac{1}{2}(10-1) = 4.5$. Hence for j up to 4 the terms are increasing, from $j = 5$ on, the terms decrease. For $n = 11$, j must be less than $\frac{1}{2}(11-1) = 5$. For $j = 5$, $(n-j)/(j+1) = 1$. Hence, up to $j = 5$ the terms increase, then $\binom{11}{5} = \binom{11}{6}$, and then the terms decrease.

EXERCISES

1. Extend the Pascal triangle to $n = 16$. Save the result for later use.

2. Prove that

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n,$$

using the fact that a set with n elements has 2^n subsets.

3. For a set of ten elements prove that there are more subsets with five elements than there are subsets with any other fixed number of elements.

4. Using the fact that $\binom{n}{r+1} = \frac{n-r}{r+1} \cdot \binom{n}{r}$, compute $\binom{30}{s}$ for $s = 1$, 2, 3, 4 from the fact that $\binom{30}{0} = 1$. [*Ans.* 30; 435; 4060; 27,405.]

5. There are $\binom{52}{13}$ different possible bridge hands. Assume that a list is made showing all these hands, and that in this list the first card in every hand is crossed out. This leaves us with a list of twelve-card hands. Prove that at least two hands in the latter list contain exactly the same cards.

6. Prove that

$$\binom{n+1}{j} = \binom{n}{j-1} + \binom{n}{j},$$

using only the fact that

$$\binom{n}{j} = \frac{n!}{j!\,(n-j)!}.$$

7. Construct a triangle in the same way that the Pascal triangle was constructed, except that whenever you add two numbers, use the addition table in Chapter II, Figure 11a. Construct the triangle for 16 rows. What does this triangle tell you about the numbers in the Pascal triangle? Use this result to check your triangle in Exercise 1.

8. In the triangle obtained in Exercise 7, what property do the rows 1, 2, 4, 8, and 16 have in common? What does this say about the numbers in the corresponding rows of the Pascal triangle? What would you predict for the terms in the 32nd row of the Pascal triangle?

9. For the following table state how one row is obtained from the preceding row and give the relation of this table to the Pascal triangle.

1	1	1	1	1	1	1
1	2	3	4	5	6	7
1	3	6	10	15	21	28
1	4	10	20	35	56	84
1	5	15	35	70	126	210
1	6	21	56	126	252	462
1	7	28	84	210	462	924

10. Referring to the table in Exercise 9, number the columns starting with 0, 1, 2, . . . and number the rows starting with 1, 2, 3, ... Let $f(n, r)$ be the element in the nth column and the rth row. The table was constructed by the rule

$$f(n, r) = f(n-1, r) + f(n, r-1)$$

for $n > 0$ and $r > 1$, and $f(n, 1) = f(0, r) = 1$ for all n and r. Verify that

$$f(n, r) = \binom{n+r-1}{n}$$

satisfies these conditions and is in fact the only choice for $f(n, r)$ which will satisfy the conditions.

11. Consider a set $\{a, a, a\}$ of three objects which cannot be distinguished from one another. Then the ordered partitions with two cells which could be distinguished are

$$[\{a, a, a\}, \varepsilon]$$
$$[\{a, a\}, \{a\}]$$
$$[\{a\}, \{a, a\}]$$
$$[\varepsilon, \{a, a, a\}].$$

List all such ordered partitions with three cells. How many are there?

[*Ans.* 10.]

12. Let $f(n, r)$ be the number of distinguishable ordered partitions with r cells which can be formed from a set of n indistinguishable objects. Show that $f(n, r)$ satisfies the conditions

$$f(n, r) = f(n - 1, r) + f(n, r - 1)$$

for $n > 0$ and $r > 1$, and $f(n, 1) = f(0, r) = 1$ for all n and r. [*Hint:* Show that $f(n, r - 1)$ is the number of partitions which have the last cell empty and $f(n - 1, r)$ is the number which have at least one element in the last cell.]

13. Using the results of Exercises 10 and 12, show that the number of distinguishable ordered partitions with r cells which can be formed from a set of n indistinguishable objects is

$$\binom{n + r - 1}{n}.$$

14. Assume that a mailman has seven letters to put in three mail boxes. How many ways can this be done if the letters are not distinguished?

[*Ans.* 36.]

15. For $n \geq r \geq k \geq s$ show that the identity

$$\binom{n}{r}\binom{r}{k}\binom{k}{s} = \binom{n}{s}\binom{n - s}{k - s}\binom{n - k}{r - k}$$

holds by replacing each binomial coefficient by a ratio of factorials.

16. Establish the identity in Exercise 15 in another way by showing that the two sides of the expression are simply two different ways of counting the number of solutions to the following problem: From a set of n people a subset of r is to be chosen; from the set of r people a subset of k is to be chosen; and from the set of k people a subset of s people is to be chosen.

17. Generalize the identity in Exercises 15 and 16 to solve the problem of finding the number of ways of selecting a t-element subset from an s-element subset from a k-element subset from an r-element subset of an n-element set, where $n \geq r \geq k \geq s \geq t$.

6. *BINOMIAL AND MULTINOMIAL THEOREMS*

It is sometimes necessary to expand products of the form $(x + y)^3$, $(x + 2y + 11z)^5$, etc. In this section we shall consider systematic ways of carrying out such expansions.

Consider first the special case $(x + y)^3$. We write this as

$$(x + y)^3 = (x + y)(x + y)(x + y).$$

To perform the multiplication, we choose either an x or a y from each of the three factors and multiply our choices together; we do this for all possible choices and add the results. We represent a particular set of choices by a two-cell partition of the numbers 1, 2, 3. In the first cell we put the numbers which correspond to factors from which we chose an x. In the second cell we put the numbers which correspond to factors from which we chose a y. For example, the partitions $[\{1, 3\}, \{2\}]$ correspond to a choice of x from the first and third factors and y from the second. The product so obtained is $xyx = x^2y$. The coefficient of x^2y in the expansion of $(x + y)^3$ will be the number of partitions which lead to a choice of two x's and one y, that is, the number of two-cell partitions of three elements with two elements in the first cell and one in the second, which is $\binom{3}{2} = 3$. More generally, the coefficient of the term of the form x^jy^{3-j} will be $\binom{3}{j}$ for $j = 0, 1, 2, 3$. Thus we can write the desired expansion as

$$\begin{aligned}(x + y)^3 &= \binom{3}{3} x^3 + \binom{3}{2} x^2y + \binom{3}{1} xy^2 + \binom{3}{0} y^3 \\ &= x^3 + 3x^2y + 3xy^2 + y^3.\end{aligned}$$

The same argument carried out for the expansion $(x + y)^n$ leads to the binomial theorem of algebra.

Binomial theorem. The expansion of $(x + y)^n$ is given by

$$(x + y)^n = x^n + \binom{n}{n-1} x^{n-1}y + \binom{n}{n-2} x^{n-2}y^2 + \ldots + \binom{n}{1} xy^{n-1} + y^n.$$

Example 1. Let us find the expansion for $(a - 2b)^3$. To fit this into the binomial theorem, we think of x as being a and y as being $-2b$. Then we have

$$\begin{aligned}(a - 2b)^3 &= a^3 + 3a^2(-2b) + 3a(-2b)^2 + (-2b)^3 \\ &= a^3 - 6a^2b + 12ab^2 - 8b^3.\end{aligned}$$

We turn now to the problem of expanding the trinomial $(x + y + z)^3$. Again we write

$$(x + y + z)^3 = (x + y + z)(x + y + z)(x + y + z).$$

This time we choose either an x or y or z from each of the three factors. Our choice is now represented by a three-cell partition of the set of numbers $\{1, 2, 3\}$. The first cell has the numbers corresponding to factors from which we choose an x, the second cell the numbers corresponding to factors from which we choose a y, and the third those from which we choose a z. For example, the partition $[\{1, 3\}, \mathcal{E}, \{2\}]$ corresponds to a choice of x from the first and third factors, no y's, and a z from the second factor. The term obtained is $xzx = x^2z$. The coefficient of the term x^2z in the expansion is thus the number of three-cell partitions with two elements in the first cell, none in the second, and one in the third. There are $\binom{3}{2,\,0,\,1} = 3$ such partitions. In general, the coefficient of the term of the form $x^ay^bz^c$ in the expansion of $(x + y + z)^3$ will be

$$\binom{3}{a,\,b,\,c} = \frac{3!}{a!\,b!\,c!}.$$

Finding this way the coefficient for each possible a, b, and c, we obtain

$$\begin{aligned}(x + y + z)^3 = x^3 + y^3 + z^3 + 3x^2y + 3xy^2 \\ + 3yz^2 + 3y^2z + 3xz^2 + 3x^2z + 6xyz.\end{aligned}$$

The same method can be carried out in general for finding the expansion of $(x_1 + x_2 + \ldots + x_r)^n$. From each factor we choose either an x_1, or x_2, or $x_3, \ldots,$ or x_r, form the product and add these products for all n possible choices. We will have r^n products, but many will be equal. A particular choice of one term from each factor determines an r-cell partition of the numbers from 1 to n. In the first cell we put the numbers of the factors from which we choose an x_1, in the second cell those from which we choose x_2, etc. A particular choice gives us a term of the form $x_1^{n_1}x_2^{n_2} \ldots x_r^{n_r}$ with $n_1 + n_2 + \ldots + n_r = n$. The corresponding partition has n_1 elements in the first cell, n_2 in the second, etc. For each such partition we obtain one such term. Hence the number of these terms which we obtain is the number of such partitions, which is

$$\binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1!\, n_2! \ldots n_r!}.$$

Thus we have the multinomial theorem.

Multinomial theorem. The expansion of $(x_1 + x_2 + \ldots + x_r)^n$ is found by adding all terms of the form

$$\binom{n}{n_1, n_2, \ldots, n_r} x_1^{n_1} x_2^{n_2} \ldots x_r^{n_r}$$

where $n_1 + n_2 + \ldots + n_r = n$.

EXERCISES

1. Expand by the binomial theorem
(a) $(x + y)^4$.
(b) $(1 + x)^5$.
(c) $(x - y)^3$.
(d) $(2x + a)^4$.
(e) $(2x - 3y)^3$.
(f) $(100 - 1)^5$.

2. Expand
(a) $(x + y + z)^4$.
(b) $(2x + y - z)^3$.
(c) $(2 + 2 + 1)^3$. (Evaluate two ways.)

3. (a) Find the coefficient of the term $x^2y^3z^2$ in the expansion of $(x + y + z)^7$. [*Ans.* 210.]
(b) Find the coefficient of the term $x^6y^3z^2$ in the expression $(x - 2y + 5z)^{11}$. [*Ans.* −924,000.]

4. Using the binomial theorem prove that

(a) $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n} = 2^n$.

(b) $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \ldots \pm \binom{n}{n} = 0$ for $n > 0$.

5. Using an argument similar to the one in Section 6, prove that

$$\binom{n+1}{i,j,k} = \binom{n}{i-1,j,k} + \binom{n}{i,j-1,k} + \binom{n}{i,j,k-1}.$$

6. Let $f(n, r)$ be the number of terms in the multinomial expansion of

$$(x_1 + x_2 + \ldots + x_r)^n$$

and show that

$$f(n, r) = \binom{n+r-1}{n}.$$

[*Hint:* Show that the conditions of Section 5, Exercise 10 are satisfied by showing that $f(n, r-1)$ is the number of terms which do not have x_r and $f(n-1, r)$ is the number which do.]

7. How many terms are there in each of the expansions:

(a) $(x + y + z)^6$? [*Ans.* 28.]

(b) $(a + 2b + 5c + d)^4$? [*Ans.* 35.]

(c) $(r + s + t + u + v)^6$? [*Ans.* 210.]

8. Prove that k^n is the sum of the numbers $\binom{n}{r_1, r_2, \ldots, r_k}$ for all choices of $r_1, r_2, \ldots, r_k$ such that $r_1 + r_2 + \ldots + r_k = n$.

SUPPLEMENTARY EXERCISES

9. Show that the problem given in Exercise 15(b) of Section 4 can also be solved by a multinomial coefficient, and hence show that

$$\binom{n}{n-r, r-k, k} = \binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}.$$

10. Show that the problem given in Exercise 16 of Section 5 can also be solved by a multinomial coefficient, and hence show that

$$\binom{n}{n-r, r-k, k-s, s} = \binom{n}{r}\binom{r}{k}\binom{k}{s} = \binom{n}{s}\binom{n-s}{k-s}\binom{n-k}{r-k}.$$

11. If $a + b + c = n$, show that

$$\binom{n}{a, b, c} = \binom{n}{a}\binom{n-a}{b}.$$

12. If $a + b + c + d = n$, show that

$$\binom{n}{a, b, c, d} = \binom{n}{a}\binom{n-a}{b}\binom{n-a-b}{c}.$$

13. If $n_1 + n_2 + \ldots + n_r = n$, guess a formula that relates the multinomial coefficient

$$\binom{n}{n_1, n_2, \ldots, n_r}$$

to a product of binomial coefficients. [*Hint:* Use the formulas in Exercises 11 and 12 to guide you.]

14. Use Exercises 11–13 to show that the multinomial coefficients can always be obtained by taking products of suitable numbers in the first n rows of the Pascal triangle.

*7. VOTING POWER

We return to the problem raised in Section 6 of Chapter II. Now we are interested not only in coalitions, but also in the power of individual members. We will develop a numerical measure of voting power that was suggested by L. S. Shapley and M. Shubik. While the measure will be explained in detail below, for the reasons for choosing this particular measure the reader is referred to the original paper.

First of all we must realize that the number of votes a man controls is not in itself a good measure of his power. If x has three votes and y has one vote, it does not necessarily follow that x has three times the power that y has. Thus if the committee has just three members {x, y, z} and z also has only one vote, then x is a dictator and y is powerless.

The basic idea of the power index is found in considering various alignments of the committee members on a number of issues. The n members are ordered $x_1, x_2, \ldots, x_n$ according to how likely they are to vote for the measure. If the measure is to carry, we must persuade x_1 and x_2 up to x_i to vote for it until we have a winning coalition. If $\{x_1, x_2, \ldots, x_i\}$ is a winning coalition but $\{x_1, x_2, \ldots, x_{i-1}\}$ is not winning, then x_i is the crucial member of the coalition. We must persuade him to vote for the measure, and he is the one hardest to persuade of the i necessary members. We call x_i the *pivot*.

For a purely mathematical measure of the power of a member we do not consider the views of the members. Rather we consider all possible ways that the members could be aligned on an issue, and see how often a given member would be the pivot. That means considering all permutations, and there will be $n!$ of them. In each permutation one member will be the pivot. The frequency with which a man is the pivot of an alignment is a good measure of his voting power.

DEFINITION. The voting power of a member of a committee is the number of alignments in which he is pivotal divided by the total number of alignments. (The total number of alignments, of course, is $n!$ for a committee of n members.)

Example 1. If all n members have one vote each, and it takes a majority vote to carry a measure, it is easy to see (by symmetry) that each member is pivot in $1/n$ of the alignments. Hence each member has power $= 1/n$. Let us illustrate this for $n = 3$. There are $3! = 6$ alignments. It takes two votes to carry a measure; hence the second member is always the pivot. The alignments are: 1**2**3, 1**3**2, 2**1**3, 2**3**1, 3**1**2, 3**2**1. The pivots are in **boldface.** Each member is pivot twice, hence has power $\frac{2}{6} = \frac{1}{3}$.

Example 2. Reconsider Chapter II, Section 6, Example 3 from this point of view. There are 24 permutations of the four members. We will list them, with the pivot in **boldface:**

wxy**z**	wx**z**y	wyx**z**	wy**z**x	w**z**xy	w**z**yx
xwy**z**	xw**z**y	xyw**z**	xy**z**w	xz**w**y	xz**y**w
yxw**z**	yx**z**w	ywx**z**	yw**z**x	yz**x**w	yz**w**x
zx**y**w	zx**w**y	zy**x**w	zy**w**x	z**w**xy	z**w**yx

We see that z has power of $\frac{14}{24}$, w has $\frac{6}{24}$, x and y have $\frac{2}{24}$ each. (Or, simplified, they have $\frac{7}{12}$, $\frac{3}{12}$, $\frac{1}{12}$, $\frac{1}{12}$ power, respectively.) We note that these ratios are much further apart than the ratio of votes which is 3:2:1:1. Here three votes are worth seven times as much as the single vote and more than twice as much as two votes.

Example 3. Reconsider Chapter II, Section 6, Example 4. By an analysis similar to the ones used so far (but too long to be included here) it can be shown that in the Security Council of the United Nations each of the Big Five has $\frac{76}{385}$ or approximately .197 power, while each of the small nations has approximately .002 power. This reproduces our intuitive feeling that, while the small nations in the Security Council are not powerless, nearly all the power is in the hands of the Big Five.

The voting powers according to the 1966 revision will be worked out in the exercises.

Example 4. In a committee of five each member has one vote, but the chairman has veto power. Hence the minimal winning coalitions are three-member coalitions including the chairman. There are $5! = 120$ permutations. The pivot cannot come before the chairman, since without the chairman we do not have a winning coalition. Hence, when the chairman is in place number 3, 4, or 5, he is the pivot. This happens in $\frac{3}{5}$ of the permutations. When he is in position 1 or 2, then the number 3 man is pivot. The number of permutations in which the chairman is in one of the first two positions and a given man is third is $2 \cdot 3! = 12$. Hence the chairman has power $\frac{3}{5}$, and each of the others has power $\frac{1}{10}$.

EXERCISES

1. A committee of three makes decisions by majority vote. Write out all permutations, and calculate the voting powers if the three members have

(a) One vote each. [*Ans.* $\frac{1}{3}$, $\frac{1}{3}$, $\frac{1}{3}$.]

(b) One vote for two of them, two votes for the third. [*Ans.* $\frac{1}{6}$, $\frac{1}{6}$, $\frac{2}{3}$.]

(c) One vote for two of them, three votes for the third. [*Ans.* 0, 0, 1.]

(d) One, two, and three votes, respectively. [*Ans.* $\frac{1}{6}$, $\frac{1}{6}$, $\frac{2}{3}$.]

(e) Two votes each for two of them, and three votes for the third. [*Ans.* $\frac{1}{3}$, $\frac{1}{3}$, $\frac{1}{3}$.]

2. Prove that in any decision-making body the sum of the powers of the members is 1.

3. What is the power of a dictator? What is the power of a "powerless" member? Prove that your answers are correct.

4. A large company issued 100,000 shares. These are held by three stockholders, who have 50,000, 49,999, and one share, respectively. Calculate the powers of the three members. [*Ans.* $\frac{2}{3}$, $\frac{1}{6}$, $\frac{1}{6}$.]

5. A committee consists of 100 members having one vote each, plus a chairman who can break ties. Calculate the power distribution. (Do *not* try to write out all permutations!)

6. In Exercise 5, give the chairman a veto instead of the power to break ties. How does this change the power distribution?

[*Ans.* The chairman has power $\frac{50}{101}$.]

7. How are the powers in Exercise 1 changed if the committee requires a $\frac{2}{3}$ vote to carry a measure?

8. If in a committee of five, requiring majority decisions, each member has one vote, then each has power $\frac{1}{5}$. Now let us suppose that two members team up, and always vote the same way. Does this increase their power? (The best way to represent this situation is by allowing only those permutations in which these two members are next to each other.)

[*Ans.* Yes, the pair's power increases from .4 to .5.]

9. If the minimal winning coalitions are known, show that the power of each member can be determined without knowing anything about the number of votes that each member controls.

10. Answer the following questions for a three-man committee.
- (a) Find all possible sets of minimal winning coalitions.
- (b) For each set of minimal winning coalitions find the distribution of voting power.
- (c) Verify that the various distributions of power found in Exercises 1 and 7 are the only ones possible.

11. In Exercise 1, parts (a) and (e) have the same answer, and parts (b) and (d) and Exercise 4 also have the same answer. Use the results of Exercise 9 to find a reason for these coincidences.

12. Compute the voting power of one of the Big Five in the Security Council of the United Nations as follows:
- (a) Show that for the nation to be pivotal it must be in the number 7 spot or later.
- (b) Show that there are $\binom{6}{2}$ 6! 4! permutations in which the nation is pivotal in the number 7 spot.
- (c) Find similar formulas for the number of permutations in which it is pivotal in the number 8, 9, 10, or 11 spot.
- (d) Use this information to find the total number of permutations in which it is pivotal, and from this compute the power of the nation.

13. Apply the method of Exercise 12 to the revised voting scheme in the Security Council (10 small-nation members, and 9 votes required to carry a measure). What is the power of a large nation? Has the power of one of the small nations increased or decreased?

[*Ans.* $\frac{421}{2145}$ (nearly the same as before); decreased.]

*8. *TECHNIQUES FOR COUNTING*

We know that there is no single method or formula for solving all counting problems. There are, however, some useful techniques that can be learned. In this section we shall discuss two problems that illustrate important techniques.

The first problem illustrates the importance of looking for a general pattern in the examination of special cases. We have seen in Section 2 of this chapter, and Exercise 6 of that section, that the following formulas hold for the number of elements in the union of two and three sets, respectively.

$$n(A_1 \cup A_2) = n(A_1) + n(A_2) - n(A_1 \cap A_2)$$

$$n(A_1 \cup A_2 \cup A_3) = n(A_1) + n(A_2) + n(A_3) - n(A_1 \cap A_2) - n(A_1 \cap A_3) - n(A_2 \cap A_3) + n(A_1 \cap A_2 \cap A_3).$$

On the basis of these formulas we might conjecture that the number of elements in the union of any finite number of sets could be obtained by adding the numbers in each of the sets, then subtracting the numbers in each possible intersection of two sets, then adding the numbers in each possible intersection of three sets, etc. If this is correct, the formula for the intersection of four sets should be

$$\begin{aligned} (1) \quad & n(A_1 \cup A_2 \cup A_3 \cup A_4) \\ &= n(A_1) + n(A_2) + n(A_3) + n(A_4) - n(A_1 \cap A_2) - n(A_1 \cap A_3) \\ &\quad - n(A_1 \cap A_4) - n(A_2 \cap A_3) - n(A_2 \cap A_4) - n(A_3 \cap A_4) \\ &\quad + n(A_1 \cap A_2 \cap A_3) + n(A_1 \cap A_2 \cap A_4) + n(A_1 \cap A_3 \cap A_4) \\ &\quad + n(A_2 \cap A_3 \cap A_4) - n(A_1 \cap A_2 \cap A_3 \cap A_4). \end{aligned}$$

Let us try to establish this formula. We must show that if u is an element of at least one of the four sets, then it is counted exactly once on the right-hand side of (1). We consider separately the cases where u is in exactly 1 of the sets, exactly 2 of the sets, etc.

For instance, if u is in exactly two of the sets it will be counted twice in the terms of the right-hand side of (1) that involve single sets, once in the terms that involve the intersection of two sets, and not at all in the terms that involve the intersections of three or four sets. Again, if u is in exactly three of the sets it will be counted three times in the terms involving single sets, twice in the terms involving intersections of two sets, once in the terms involving the intersections of three sets, and not at all in the last term involving the intersection of all four sets. Considering each possibility we have the following table.

Number of sets that contain u	Number of times it is counted
1	1
2	$2 - 1$
3	$3 - 3 + 1$
4	$4 - 6 + 4 - 1$

We see from this that, in every case, u is counted exactly once on the right-hand side of (1). Furthermore, if we look closely, we detect a pattern in the numbers in the right-hand column of the above table. If we put a -1 in front of these numbers we have

1	$-1+1$
2	$-1+2-1$
3	$-1+3+3-1$
4	$-1+4-6+4-1$

We now recognize that these numbers are simply the numbers in the first four rows of the Pascal triangle, but with alternating + and − signs. Since we put a -1 in each row of the table, we want to show that the sum of each row is 0. If that is true, it should be a general property of the Pascal triangle. That is, if we put alternating signs in the jth row of the Pascal triangle, we should get a sum of 0. But this is indeed the case, since, by the binomial theorem, for $j > 0$,

$$0 = \pm(1-1)^j = 1 - \binom{j}{1} + \binom{j}{2} - \binom{j}{3} + \ldots \pm 1$$

$$= -1 + \binom{j}{1} - \binom{j}{2} + \binom{j}{3} - \ldots \mp 1.$$

Thus we have not only seen why the formula works for the case of four sets, but we have also found the method for proving the formula for the general case. That is, suppose we wish to establish that the number of elements in the union of n sets may be obtained as an alternating sum by adding the numbers of elements in each of the sets, subtracting the numbers of elements in each pairwise intersection of the sets, adding the numbers of elements in each intersection of three sets, etc. Consider an element u that is in exactly j of the sets. Let us see how many times u will be counted in the alternating sum. If it is in j of the sets, it will first be counted j times in the sum of the elements in the sets by themselves. For u to be in the intersection of two sets, we must choose two of the j sets to which it belongs. This can be done in $\binom{j}{2}$ different ways. Hence an amount $\binom{j}{2}$ will be subtracted from the sum. To be in the intersection of three sets, we must choose three of the j sets containing u. This can be done in $\binom{j}{3}$ different ways. Thus, an amount $\binom{j}{3}$ will be added to the sum, etc. Hence the total number of times u will be counted by the alternating sum is

$$\binom{j}{1} - \binom{j}{2} + \binom{j}{3} - \ldots \pm 1.$$

Since we have just seen that, if we add -1 to the sum, we obtain 0. Hence the sum itself must always be 1. That is, no matter how many sets u is in, it will be counted exactly once by the alternating sum, and this is true for every element u in the union. We have thus established the general formula

$$\begin{aligned}(2)\qquad & n(A_1 \cup A_2 \cup \ldots \cup A_n) \\ &= n(A_1) + n(A_2) + \ldots + n(A_n) \\ &\quad - n(A_1 \cap A_2) - n(A_1 \cap A_3) - \ldots \\ &\quad + n(A_1 \cap A_2 \cap A_3) + n(A_1 \cap A_2 \cap A_4) + \ldots \\ &\quad + \ldots \pm n(A_1 \cap A_2 \cap \ldots \cap A_n).\end{aligned}$$

This formula is called the *inclusion-exclusion formula* for the number of elements in the union of sets. It can be extended to formulas for counting the number of elements that occur in two of the sets, three of the sets, etc. See Exercises 21, 25, and 27.

Example 1. In a high school the following language enrollments are recorded for the senior class.

English	150
French	75
German	35
Spanish	50

Also, the following overlaps are noted.

Taking English and French	70
Taking English and German	30
Taking English and Spanish	40
Taking French and German	5
Taking English, French and German	2

If every student takes at least one language, how many seniors are there?

Let E, F, G, and S be the sets of students taking English, French, German, and Spanish, respectively. Using formula (1) and ignoring empty sets, we have

$$\begin{aligned} n(E \cup F \cup G \cup S) & \\ &= n(E) + n(F) + n(G) + n(S) - n(E \cap F) - n(E \cap G) \\ &\qquad - n(E \cap S) - n(F \cap G) + n(E \cap F \cap G) \\ &= 150 + 75 + 35 + 50 - 70 - 30 - 40 - 5 + 2 \\ &= 167. \end{aligned}$$

Since every student takes at least one language, the total number of students is 167.

Example 2. The four words

TABLE, BASIN, CLASP, BLUSH

have the following interesting properties. Each word consists of five different letters. Any two words have exactly two letters in common. Any three words have one letter in common. No letter occurs in all four words. How many different letters are there?

Letting the words be sets of letters, there are $\binom{4}{1}$ ways of taking these sets one at a time, $\binom{4}{2}$ ways of taking them two at a time, etc. Hence formula (2) gives

$$\binom{4}{1}\cdot 5 - \binom{4}{2}\cdot 2 + \binom{4}{3}\cdot 1 - \binom{4}{4}\cdot 0 = 12$$

as the number of distinct letters. The reader should verify this answer by direct count.

It often happens that a counting problem can be formulated in a number of different ways that sound quite different but that are in fact equivalent. And in one of these ways the answer may suggest itself readily. To illustrate how a reformulation can make a hard sounding problem easy, we give an alternate method for solving the problem considered in Exercises 9–13 of Section 5.

The problem is to count the number of ways that n indistinguishable objects can be put into r cells. For instance, if there are three objects and three cells, the number of different ways can be enumerated as follows (using O for object and bars to indicate the sides of the cells):

```
|OOO|   |   |
|OO |O  |   |
|OO |   |O  |
|O  |OO |   |
|O  |O  |O  |
|O  |   |OO |
|   |OOO|   |
|   |OO |O  |
|   |O  |OO |
|   |   |OOO|
```

We see that in this case there are ten ways the task can be accomplished. But the answer for the general case is not clear.

If we look at the problem in a slightly different manner, the answer suggests itself. Instead of putting the objects *in* the cells, we imagine putting the cells *around* the objects. In the above case we see that three cells are constructed from four bars. Two of these bars must be placed at the ends. The two other bars together with the three objects we regard as occupying five intermediate positions. Of these five intermediate positions we must choose two of them for bars and three for the objects. Hence the total number of ways we can accomplish the task is $\binom{5}{2} = \binom{5}{3} = 10$, which is the answer we got by counting all the ways.

For the general case we can argue in the same manner. We have r cells and n objects. We need $r + 1$ bars to form the r cells, but two of these must be fixed on the ends. The remaining $r - 1$ bars together with the n objects occupy $r - 1 + n$ intermediate positions. And we must choose $r - 1$ of these for the bars and the remaining n for the objects. Hence our task can be accomplished in

$$\binom{n + r - 1}{r - 1} = \binom{n + r - 1}{n}$$

different ways.

Example 3. Seven people enter an elevator that will stop at five floors. In how many different ways can the people leave the elevator if we are interested only in the number that depart at each floor, and do not distinguish among the people? According to our general formula, the answer is

$$\binom{7+5-1}{7} = \binom{11}{7} = 330.$$

Suppose we are interested in finding the number of such possibilities in which at least one person gets off at each floor. We can then arbitrarily assign one person to get off at each floor, and the remaining two can get off at any floor. They can get off the elevator in

$$\binom{2+5-1}{2} = \binom{6}{2} = 15$$

different ways.

EXERCISES

1. The survey discussed in Exercise 8 of Section 2 has been enlarged to include a fourth magazine D. It was found that no one who reads either magazine A or magazine B reads magazine D. However, 10 per cent of the people read magazine D and 5 per cent read both C and D. What per cent of the people do not read any magazine? [*Ans.* 5 per cent.]

2. A certain college administers three qualifying tests. They announce the following results: "Of the students taking the tests, 2 per cent failed all three tests, 6 per cent failed tests A and B, 5 per cent failed A and C, 8 per cent failed B and C, 29 per cent failed test A, 32 per cent failed B, and 16 per cent failed C." How many students passed all three qualifying tests?

3. Four partners in a game require a score of exactly 20 points to win. In how many ways can they accomplish this? [*Ans.* $\binom{23}{3}$.]

4. In how many ways can eight apples be distributed among four boys? In how many ways can this be done if each boy is to get at least one apple?

5. Suppose we have n balls and r boxes with $n \geq r$. Show that the number of different ways that the balls can be put into the boxes which insures that there is at least one ball in every box is $\binom{n-1}{r-1}$.

6. Identical prizes are to be distributed among five boys. It is observed that there are 15 ways that this can be done if each boy is to get at least one prize. How many prizes are there? [*Ans.* 7.]

7. Let $p_1, p_2, \ldots, p_n$ be n statements relative to a possibility space $\mathcal{U}$. Show that the inclusion-exclusion formula gives a formula for the number of elements in the truth set of the disjunction $p_1 \vee p_2 \vee \ldots \vee p_n$ in terms of the numbers of elements in the truth sets of conjunctions formed from subsets of these statements.

8. A man asks his secretary to put letters written to seven different persons into addressed envelopes. Find the number of ways that this can be done so that at least one person gets his own letter. [*Hint:* Use the result of Exercise 7, letting p_i be the statement "The ith man gets his own letter."]

[*Ans.* 3186.]

9. Consider the numbers from 2 to 10 inclusive. Let A_2 be the set of numbers divisible by 2 and A_3 the set of numbers divisible by 3. Find $n(A_2 \cup A_3)$ by using the inclusion-exclusion formula. From this result find the number of prime numbers between 2 and 10 (where a prime number is a number divisible only by itself and by 1). [*Hint:* Be sure to count the numbers 2 and 3 among the primes.]

10. Use the method of Exercise 9 to find the number of prime numbers between 2 and 100 inclusive. [*Hint:* Consider first the sets A_2, A_3, A_5, and A_7.]

[*Ans.* 25.]

11. Verify that the following formula gives the number of elements in the intersection of three sets.

$$n(A_1 \cap A_2 \cap A_3) = n(A_1) + n(A_2) + n(A_3) - n(A_1 \cup A_2) - n(A_1 \cup A_3) - n(A_2 \cup A_3) + n(A_1 \cup A_2 \cup A_3).$$

12. Show that if we replace $\cap$ by $\cup$ and $\cup$ by $\cap$ in formula (2), we get a valid formula for the number of elements in the intersection of n sets. [*Hint:* Apply the inclusion-exclusion formula to the left-hand side of

$$n[\tilde{A}_1 \cup \tilde{A}_2 \cup \ldots \cup \tilde{A}_m) = n(\mathcal{U}) - n(A_1 \cap A_2 \cap \ldots \cap A_m).]$$

13. For $n \leq m$ prove that

$$\binom{m}{0}\binom{n}{0} + \binom{m}{1}\binom{n}{1} + \binom{m}{2}\binom{n}{2} + \ldots + \binom{m}{n}\binom{n}{n} = \binom{m+n}{n}$$

by carrying out the following two steps:

(a) Show that the left-hand side counts the number of ways of choosing equal numbers of men and women from sets of m men and n women.

(b) Show that the right-hand side also counts the same number by showing that we can select equal numbers of men and women by selecting any subset of n persons from the whole set, and then combining the men selected with the women not selected.

14. By an ordered partition of n with r elements we mean a sequence of nonnegative integers, possibly some 0, written in a definite order, and having n as their sum. For instance, $\{1, 0, 3\}$ and $\{3, 0, 1\}$ are two different ordered partitions of 4 with three elements. Show that the number of ordered partitions of n with r elements is $\binom{n+r-1}{n}$.

15. Show that the number of different possibilities for the outcomes of rolling n dice is $\binom{n+5}{n}$.

Note: Exercises 16–19 illustrate an important counting technique called the *reflection principle.* In Figure 6 we show a path from the point (0, 2) to

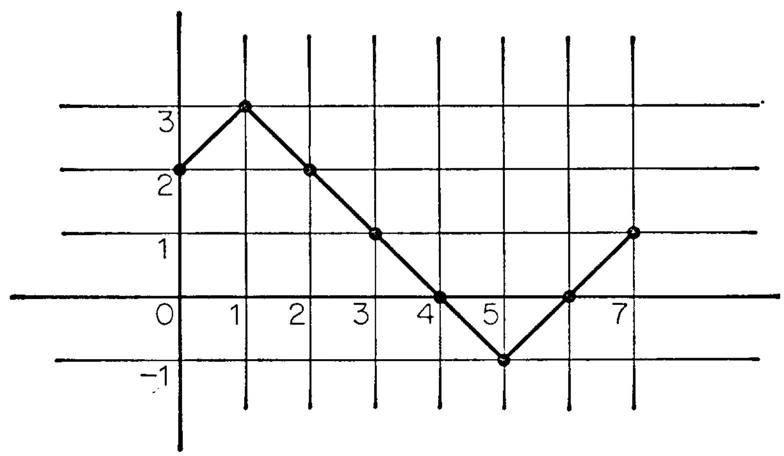

Figure 6

the point (7, 1). We shall be interested in counting the number of paths of this type where at each step the path moves one unit to the right, and either one unit *up* or one unit *down.* We shall see that this model is useful for analyzing voting outcomes.

16. Show that the number of different paths leading from the point (0, 2) to (7, 1) is $\binom{7}{3}$. [*Hint:* Seven decisions must be made, of which three moves are up and the rest down.]

17. Show that the number of different paths from (0, 2) to (7, 1) which touch the x-axis at least once is the same as the total number of paths from the point $(0, -2)$ to the point (7, 1). [*Hint:* Show that for every path to be counted from (0, 2) that touches the x-axis, there corresponds a path from $(0, -2)$ to (7, 1) obtained by reflecting the part of the path to the first touching point through the x-axis. A specific example is shown in Figure 7.]

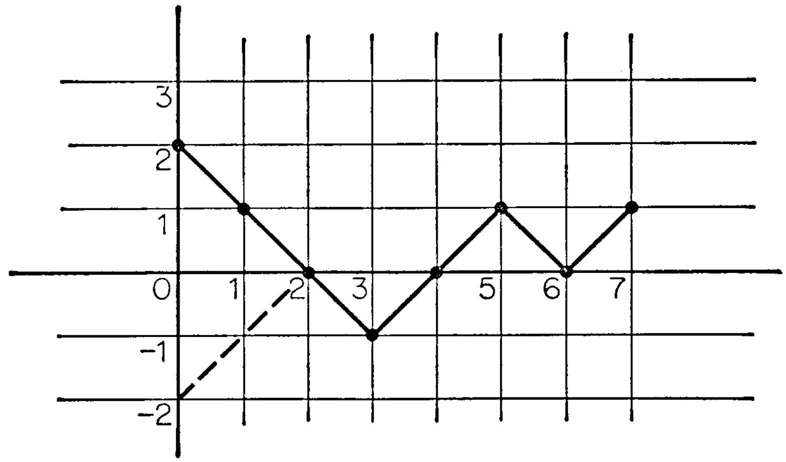

Figure 7

18. Use the results of Exercises 16 and 17 to find the number of paths from (0, 2) to (7, 1) that never touch the x-axis. [*Ans.* 14.]

19. Nine votes are cast in a race between two candidates A and B. Candidate A wins by one vote. Find the number of ways the ballots can be counted so that candidate A is leading throughout the entire count. [*Hint:* The first vote counted must be for A. Counting the remaining eight votes corresponds to a path from (1, 1) to (9, 1). We want the number of paths that never touch the x-axis.] [*Ans.* 14.]

20. Let the symbol $n_r^{(k)}$ stand for "the number of elements that are in k or more of the r sets $A_1, A_2, \ldots, A_r$." Show that $n_3^{(1)} = n(A_1 \cup A_2 \cup A_3)$.

21. Show that

$$\begin{aligned} n_3^{(2)} &= n((A_1 \cap A_2) \cup (A_1 \cap A_3) \cup (A_2 \cap A_3)) \\ &= n(A_1 \cap A_2) + n(A_1 \cap A_3) + n(A_2 \cap A_3) - 2n(A_1 \cap A_2 \cap A_3) \end{aligned}$$

by using the inclusion-exclusion formula. Also develop an independent argument for the last formula.

22. Use Exercise 21 to find the number of letters that appear two or more times in the three words TABLE, BASIN, and CLASP.

23. Give an interpretation for $n_3^{(1)} - n_3^{(2)}$.

24. Use Exercise 23 to find the number of letters that occur exactly once in the three words of Exercise 22.

25. Develop a general argument like that in Exercise 21 to show that

$$\begin{aligned} n_4^{(2)} = {} & n(A_1 \cap A_2) + n(A_1 \cap A_3) + n(A_1 \cap A_4) + n\,(A_2 \cap A_3) \\ & + n(A_2 \cap A_4) + n(A_3 \cap A_4) - 2[n(A_1 \cap A_2 \cap A_3) \\ & + n(A_1 \cap A_2 \cap A_4) + n(A_1 \cap A_3 \cap A_4) + n(A_2 \cap A_3 \cap A_4)] \\ & + 3n(A_1 \cap A_2 \cap A_3 \cap A_4). \end{aligned}$$

26. Use Exercise 25 to find the number of letters used two or more times in the four words of Example 2.

27. From the formulas in Exercises 21 and 25 guess the general formula for $n_r^{(2)}$ and develop a general argument to establish its correctness.

SUGGESTED READING

Shapley, L. S., and M. Shubik, "A Method for Evaluating the Distribution of Power in a Committee System," *The American Political Science Review*, XLVIII (1954), pp. 787–792.

Whitworth, W. A., *Choice and Chance, with 1000 Exercises*, Stechert, New York, 1934.

Goldberg, S., *Probability An Introduction*, Prentice-Hall, Englewood Cliffs, N. J., 1960.

Parzen, E., *Modern Probability Theory and Its Applications*, Wiley, New York, 1960.