II

Sets and subsets

1. INTRODUCTION

A well-defined collection of objects is known as a *set*. This concept, in its complete generality, is of great importance in mathematics since all of mathematics can be developed by starting from it.

The various pieces of furniture in a given room form a set. So do the books in a given library, or the integers between 1 and 1,000,000, or all the ideas that mankind has had, or the human beings alive between one billion B.C. and ten billion A.D. These examples are all examples of *finite* sets, that is, sets having a finite number of elements. All the sets discussed in this book will be finite sets.

There are two essentially different ways of specifying a set. One can give a rule by which it can be determined whether or not a given object is a member of the set, or one can give a complete list of the elements in the set. We shall say that the former is a *description* of the set and the latter is a *listing* of the set. For example, we can define a set of four people as (a) the members of the string quartet which played in town last night, or (b) four particular persons whose names are Jones, Smith, Brown, and Green. It is customary to use braces to surround the listing of a set; thus the set above should be listed {Jones, Smith, Brown, Green}.

We shall frequently be interested in sets of logical possibilities, since the analysis of such sets is very often a major task in the solving of a problem. Suppose, for example, that we were interested in the successes of three candidates who enter the presidential primaries (we assume there are no other entries). Suppose that the key primaries will be held in New Hampshire, Minnesota, Wisconsin, and California. Assume that candidate A enters all the primaries, that B does not contest in New Hampshire's primary, and C does not contest in Wisconsin's. A list of the logical possibilities is given in Figure 1. Since the New Hampshire and Wisconsin primaries can each end in two ways, and the Minnesota and California primaries can each end in three ways, there are in all $2 \cdot 2 \cdot 3 \cdot 3 = 36$ different logical possibilities as listed in Figure 1.

A set that consists of some members of another set is called a *subset* of that set. For example, the set of those logical possibilities in Figure 1 for which the statement "Candidate A wins at least three primaries" is true, is a subset of the set of all logical possibilities. This subset can also be defined by listing its members: {P1, P2, P3, P4, P7, P13, P19}.

In order to discuss all the subsets of a given set, let us introduce the following terminology. We shall call the original set the *universal set*, one-element subsets will be called *unit sets*, and the set which contains no members the *empty set*. We do not introduce special names for other kinds of subsets of the universal set. As an example, let the universal set $\mathcal{U}$ consist of the three elements $\{a, b, c\}$. The *proper subsets* of $\mathcal{U}$ are those sets containing some but not all of the elements of $\mathcal{U}$. The proper subsets consist of three two-element sets namely, $\{a, b\}$, $\{a, c\}$, and $\{b, c\}$ and three unit sets, namely, $\{a\}$, $\{b\}$, and $\{c\}$. To complete the picture, we also consider the universal set a subset (but not a proper subset) of itself, and we consider the empty set $\mathcal{E}$, that contains no elements of $\mathcal{U}$, as a subset of $\mathcal{U}$. At first it may seem strange that we should include the sets $\mathcal{U}$ and $\mathcal{E}$ as subsets of $\mathcal{U}$, but the reasons for their inclusion will become clear later.

We saw that the three-element set above had $8 = 2^3$ subsets. In general, a set with n elements has 2^n subsets, as can be seen in the following manner. We form subsets P of $\mathcal{U}$ by considering each of the elements of $\mathcal{U}$ in turn and deciding whether or not to include it in the subset P. If we decide to put every element of $\mathcal{U}$ into P, we get the universal set, and if we decide to put no element of $\mathcal{U}$ into P, we get the empty set. In most cases we will put some but not all the

Possibility Number	Winner in New Hampshire	Winner in Minnesota	Winner in Wisconsin	Winner in California
P1	A	A	A	A
P2	A	A	A	B
P3	A	A	A	C
P4	A	A	B	A
P5	A	A	B	B
P6	A	A	B	C
P7	A	B	A	A
P8	A	B	A	B
P9	A	B	A	C
P10	A	B	B	A
P11	A	B	B	B
P12	A	B	B	C
P13	A	C	A	A
P14	A	C	A	B
P15	A	C	A	C
P16	A	C	B	A
P17	A	C	B	B
P18	A	C	B	C
P19	C	A	A	A
P20	C	A	A	B
P21	C	A	A	C
P22	C	A	B	A
P23	C	A	B	B
P24	C	A	B	C
P25	C	B	A	A
P26	C	B	A	B
P27	C	B	A	C
P28	C	B	B	A
P29	C	B	B	B
P30	C	B	B	C
P31	C	C	A	A
P32	C	C	A	B
P33	C	C	A	C
P34	C	C	B	A
P35	C	C	B	B
P36	C	C	B	C

Figure 1

elements into P and thus obtain a proper subset of $\mathcal{U}$. We have to make n decisions, one for each element of the set, and for each decision we have to choose between two alternatives. We can make these decisions in $2 \cdot 2 \cdot \ldots \cdot 2 = 2^n$ ways, and hence this is the number of different subsets of $\mathcal{U}$ that can be formed. Observe that our formula would not have been so simple if we had not included the universal set and the empty set as subsets of $\mathcal{U}$.

In the example of the voting primaries above there are 2^{36} or about 70 billion subsets. Of course, we cannot deal with this many subsets in a practical problem, but fortunately we are usually interested in only a few of the subsets. The most interesting subsets are those which can be defined by means of a simple rule such as "the set of all logical possibilities in which C loses at least two primaries." It would be difficult to give a simple description for the subset containing the elements {P1, P4, P14, P30, P34}. On the other hand, we shall see in the next section how to define new subsets in terms of subsets already defined.

Examples. We illustrate the two different ways of specifying sets in terms of the primary voting example. Let the universal set $\mathcal{U}$ be the logical possibilities given in Figure 1.

1. What is the subset of $\mathcal{U}$ in which candidate B wins more primaries than either of the other candidates? *Answer:* {P11, P12, P17, P23, P26, P28, P29}.

2. What is the subset in which the primaries are split two and two? *Answer:* {P5, P8, P10, P15, P21, P30, P31, P35}.

3. Describe the set {P1, P4, P19, P22}. *Answer:* The set of possibilities for which A wins in Minnesota and California.

4. How can we describe the set {P18, P24, P27}? *Answer:* The set of possibilities for which C wins in California, and the other primaries are split three ways.

EXERCISES

1. In the primary example, give a listing for each of the following sets.
 (a) The set in which C wins at least two primaries.
 (b) The set in which the first three primaries are won by the same candidate.
 (c) The set in which B wins all four primaries.

2. The primaries are considered decisive if a candidate can win three primaries, or if he wins two primaries including California. List the set in which the primaries are decisive.

3. Give simple descriptions for the following sets (referring to the primary example).

(a) {P33, P36}.
(b) {P10, P11, P12, P28, P29, P30}.
(c) {P6, P20, P22}.

4. Joe, Jim, Pete, Mary, and Peg are to be photographed. They want to line up so that boys and girls alternate. List the set of all possibilities.

5. In Exercise 4, list the following subsets.

(a) The set in which Pete and Mary are next to each other.
(b) The set in which Peg is between Joe and Jim.
(c) The set in which Jim is in the middle.
(d) The set in which Mary is in the middle.
(e) The set in which a boy is at each end.

6. Pick out all pairs in Exercise 5 in which one set is a subset of the other.

7. A TV producer is planning a half-hour show. He wants to have a combination of comedy, music, and commercials. If each is allotted a multiple of five minutes, construct the set of possible distributions of time. (Consider only the total time allotted to each.)

8. In Exercise 7, list the following subsets.

(a) The set in which more time is devoted to comedy than to music.
(b) The set in which no more time is devoted to commercials than to either music or comedy.
(c) The set in which exactly five minutes is devoted to music.
(d) The set in which all three of the above conditions are satisfied.

9. In Exercise 8, find two sets, each of which is a proper subset of the set in (a) and also of the set in (c).

10. Let $\mathcal{U}$ be the set of paths in Figure 22 of Chapter I. Find the subset in which

(a) Two balls of the same color are drawn.
(b) Two different color balls are drawn.

11. A set has 101 elements. How many subsets does it have? How many of the subsets have an odd number of elements? [*Ans.* 2^{101}; 2^{100}.]

12. Do Exercise 11 for the case of a set with 102 elements.

2. *OPERATIONS ON SUBSETS*

In Chapter I we considered the ways in which one could form new statements from given statements. Now we shall consider an analogous procedure, the formation of new sets from given sets. We shall assume that each of the sets that we use in the combination is a subset of some universal set, and we shall also want the newly formed set to be a subset of the same universal set. As usual, we can specify a newly formed set either by a description or by a listing.

If P and Q are two sets, we shall define a new set $P \cap Q$, called the *intersection* of P and Q as follows: $P \cap Q$ is the set which contains those and only those elements which belong to both P and Q. As an example, consider the logical possibilities listed in Figure 1. Let P be the subset in which candidate A wins at least three primaries, i.e., the set {P1, P2, P3, P4, P7, P13, P19}; let Q be the subset in which A wins the first two primaries, i.e., the set {P1, P2, P3, P4, P5, P6}. Then the intersection $P \cap Q$ is the set in which both events take place, i.e., where A wins the first two primaries *and* wins at least three primaries. Thus $P \cap Q$ is the set {P1, P2, P3, P4}.

If P and Q are two sets, we shall define a new set $P \cup Q$ called the *union* of P and Q as follows: $P \cup Q$ is the set that contains those and only those elements that belong either to P or to Q (or to both). In the example in the paragraph above, the union $P \cup Q$ is the set of possibilities for which either A wins the first two primaries *or* wins at least three primaries, i.e., the set {P1, P2, P3, P4, P5, P6, P7, P13, P19}.

To help in visualizing these operations we shall draw diagrams, called Venn diagrams, which illustrate them. We let the universal set be a rectangle and let subsets be circles drawn inside the rectangle.

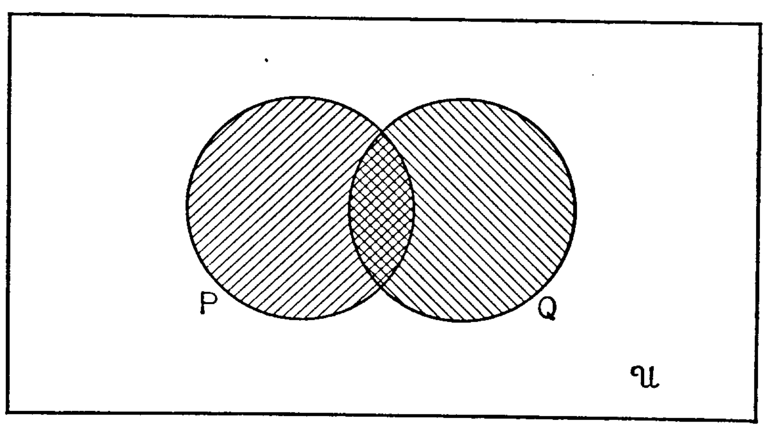


Figure 2

In Figure 2 we show two sets P and Q as shaded circles. Then the doubly crosshatched area is the intersection $P \cap Q$ and the total shaded area is the union $P \cup Q$.

If P is a given subset of the universal set $\mathcal{U}$, we can define a new set $\tilde{P}$ called the *complement* of P as follows: $\tilde{P}$ is the set of all elements of $\mathcal{U}$ that are *not* contained in P. For example, if, as above, Q is the set in which candidate A wins the first two primaries, then $\tilde{Q}$ is the set {P7, P8, . . . , P36}. The shaded area in Figure 3 is the complement of the set P. Observe that the complement of the empty set $\mathcal{E}$ is the universal set $\mathcal{U}$, and also that the complement of the universal set is the empty set.

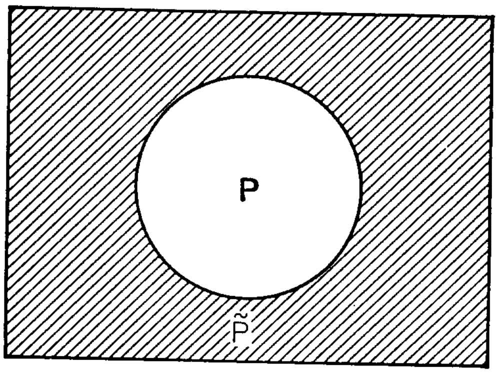


Figure 3

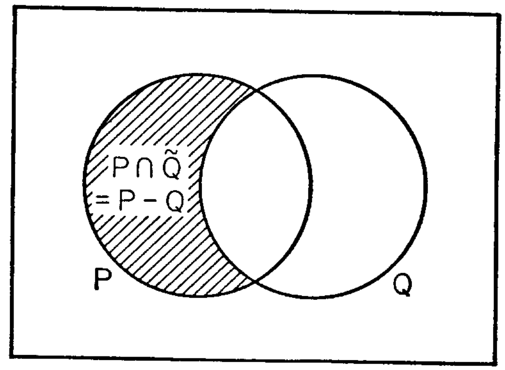


Figure 4

Sometimes we shall be interested in only part of the complement of a set. For example, we might wish to consider the part of the complement of the set Q that is contained in P, i.e., the set $P \cap \tilde{Q}$. The shaded area in Figure 4 is $P \cap \tilde{Q}$.

A somewhat more suggestive definition of this set can be given as follows: Let $P - Q$ be the *difference* of P and Q, that is, the set that contains those elements of P that do not belong to Q. Figure 4 shows that $P \cap \tilde{Q}$ and $P - Q$ are the same set. In the primary voting example above, the set $P - Q$ can be listed as {P7, P13, P19}.

The complement of a subset is a special case of a difference set, since we can write $\tilde{Q} = \mathcal{U} - Q$. If P and Q are nonempty subsets whose intersection is the empty set, i.e., $P \cap Q = \mathcal{E}$, then we say that they are *disjoint* subsets.

Examples. In the primary voting example let R be the set in which A wins the first three primaries, i.e., the set {P1, P2, P3}; let S be the set in which A wins the last two primaries, i.e., the set {P1, P7, P13, P19, P25, P31}. Then $R \cap S =$ {P1} is the set in which A wins the first three primaries and also the last two, that is, he wins all the primaries. We also have

$$R \cup S = \{P1, P2, P3, P7, P13, P19, P25, P31\},$$

which can be described as the set in which A wins the first three primaries or the last two. The set in which A does not win the first three primaries is $\tilde{R} = \{P4, P5, \ldots, P36\}$. Finally, we see that the difference set $R - S$ is the set in which A wins the first three primaries but not both of the last two. This set can be found by taking from R the element P1 which it has in common with S, so that $R - S = \{P2, P3\}$.

EXERCISES

1. Draw Venn diagrams for $P \cap Q$, $P \cap \tilde{Q}$, $\tilde{P} \cap Q$, $\tilde{P} \cap \tilde{Q}$.

2. Give a step-by-step construction of the diagram for $(\tilde{P} - Q) \cup (P \cap \tilde{Q})$.

3. Venn diagrams are also useful when three subsets are given. Construct such a diagram, given the subsets P, Q, and R. Identify each of the eight resulting areas in terms of P, Q, and R.

4. In testing blood, three types of antigens are looked for: A, B, and Rh. Every person is classified doubly. He is Rh positive if he has the Rh antigen, and Rh negative otherwise. He is type AB, A, or B depending on which of the other antigens he has, with type O having neither A nor B. Draw a Venn diagram, and identify each of the eight areas.

5. Considering only two subsets, the set X of people having antigen A, and the set Y of people having antigen B, define (symbolically) the types AB, A, B, and O.

6. A person can receive blood from another person if he has all the antigens of the donor. Describe in terms of X and Y the sets of people who can give to each of the four types. Identify these sets in terms of blood types.

7.

	Liked Very Much	Liked Slightly	Disliked Slightly	Disliked Very Much
Men	1	3	5	10
Women	6	8	3	1
Boys	5	5	3	2
Girls	8	5	1	1

This tabulation records the reaction of a number of spectators to a television show. All the categories can be defined in terms of the following four: M (male), G (grown-up), L (liked), Vm (very much). How many people fall into each of the following categories?

(a) M. [*Ans.* 34.]

(b) $\tilde{L}$.

(c) Vm.

(d) $M \cap \tilde{G} \cap \tilde{L} \cap Vm$. [*Ans.* 2.]

(e) $\tilde{M} \cap G \cap L$.

(f) $(M \cap G) \cup (L \cap Vm)$.

(g) $\widetilde{(M \cap G)}$. [*Ans.* 48.]

(h) $(\tilde{M} \cup \tilde{G})$.

(i) $(M - G)$.

(j) $[\tilde{M} - (G \cap L \cap \widetilde{Vm})]$.

8. In a survey of 100 students, the numbers studying various languages were found to be: Spanish, 28; German, 30; French, 42; Spanish and German, 8; Spanish and French, 10; German and French, 5; all three languages, 3.

(a) How many students were studying no language? [*Ans.* 20.]

(b) How many students had French as their only language? [*Ans.* 30.]

(c) How many students studied German if and only if they studied French? [*Ans.* 38.]

[*Hint:* Draw a Venn diagram with three circles, for French, German, and Spanish students. Fill in the numbers in each of the eight areas, using the data given above. Start from the end of the list and work back.]

9. In a later survey of the 100 students (see Exercise 8), numbers studying the various languages were found to be: German only, 18; German but not Spanish, 23; German and French, 8; German, 26; French, 48; French and Spanish, 8; no language, 24.

(a) How many students took Spanish? [*Ans.* 18.]

(b) How many took German and Spanish but not French? [*Ans.* None.]

(c) How many took French if and only if they did not take Spanish? [*Ans.* 50.]

10. The report of one survey of the 100 students (see Exercise 8) stated that the numbers studying the various languages were: all three languages, 5; German and Spanish, 10; French and Spanish, 8; German and French, 20; Spanish, 30; German, 23; French, 50. The surveyor who turned in this report was fired. Why?

11. A recent survey of 100 Dartmouth students has revealed the information about their dates that is summarized in the following table.

	Beautiful and Intelligent	Plain and Intelligent	Beautiful and Dumb	Plain and Dumb
Blonde	6	9	10	20
Brunette	7	11	15	9
Redhead	2	3	8	0

Let BL = blondes, BR = brunettes, R = redheads, BE = beautiful girls, D = dumb girls. Determine the number of girls in each of the following classes.

(a) $BL \cap BE \cap D$. [*Ans.* 10.]

(b) BR.

(c) $R \cap \tilde{D}$.

(d) $(BR \cup R) \cap \widetilde{(BE \cup \tilde{D})}$. [*Ans.* 46.]

(e) $\widetilde{BL} \cup (\widetilde{BE} \cap D)$.

12. In Exercise 11, which set of each of the following pairs has more girls as members?

(a) $\widetilde{(BL \cup BR)}$ or R.

(b) $D \cap \widetilde{BE}$ or $BL - (D \cap \widetilde{BE})$.

(c) $\mathcal{E}$ or $R \cap \widetilde{BE} \cap D$.

3. *THE RELATIONSHIP BETWEEN SETS AND COMPOUND STATEMENTS*

The reader may have observed several times in the preceding sections that there was a close connection between sets and statements, and between set operations and compounding operations. In this section we shall formalize these relationships.

If we have a number of statements relative to a set of logical possibilities, there is a natural way of assigning a set to each statement. First of all, we take the set of logical possibilities as our universal set. Then to each statement we assign the subset of logical possibilities of the universal set for which that statement is true. This idea is so important that we embody it in a formal definition.

DEFINITION. Let $\mathcal{U}$ be a set of logical possibilities, let p be a statement relative to it, and let P be that subset of the possibilities for which p is true; then we call P the *truth set* of p.

If p and q are statements, then $p \vee q$ and $p \wedge q$ are also statements and hence must have truth sets. To find the truth set of $p \vee q$, we observe that it is true whenever p is true or q is true (or both). Therefore we must assign to $p \vee q$ the logical possibilities which are in P or in Q (or both); that is, we must assign to $p \vee q$ the set $P \cup Q$. On the other hand, the statement $p \wedge q$ is true only when both p and q are true, so that we must assign to $p \wedge q$ the set $P \cap Q$.

Thus we see that there is a close connection between the logical operation of disjunction and the set operation of union, and also between conjunction and intersection. A careful examination of the definitions of union and intersection shows that the word "or" occurs in the definition of union and the word "and" occurs in the definition of intersection. Thus the connection between the two theories is not surprising.

Since the connective "not" occurs in the definition of the complement of a set, it is not surprising that the truth set of $\sim p$ is $\tilde{P}$. This follows since $\sim p$ is true when p is false, so that the truth set of $\sim p$ contains all logical possibilities for which p is false, that is, the truth set of $\sim p$ is $\tilde{P}$.

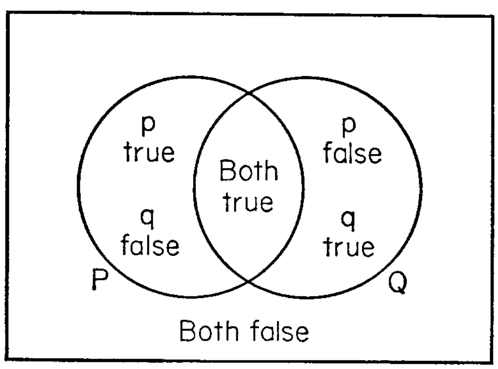


Figure 5

The truth sets of two propositions p and q are shown in Figure 5. Also marked on the diagram are the various logical possibilities for these two statements. The reader should pick out in this diagram the truth sets of the statements $p \vee q$, $p \wedge q$, $\sim p$, and $\sim q$.

The connection between a statement and its truth set makes it possible to "translate" a problem about compound statements into a problem about sets. It is also possible to go in the reverse direction. Given a problem about sets, think of the universal set as being a set of logical possibilities and think of a subset as being the truth set of a statement. Hence we can "translate" a problem about sets into a problem about compound statements.

So far we have discussed only the truth sets assigned to compound statements involving $\vee$, $\wedge$, and $\sim$. All the other connectives can be defined in terms of these three basic ones, so that we can deduce what truth sets should be assigned to them. For example, we know that $p \rightarrow q$ is equivalent to $\sim p \vee q$ (see Figure 28 of Chapter I). Hence the truth set of $p \rightarrow q$ is the same as the truth set of $\sim p \vee q$, that is, it is $\tilde{P} \cup Q$. The Venn diagram for $p \rightarrow q$ is shown in Figure 6, where the shaded area is the truth set for the statement. Observe that the unshaded area in Figure 6 is the set $P - Q = P \cap \tilde{Q}$, which is the truth set of the statement $p \wedge \sim q$. Thus the shaded area is the set $\widetilde{(P - Q)} = \widetilde{P \cap \tilde{Q}}$, which is the truth set of the statement $\sim[p \wedge \sim q]$. We have thus discovered the fact that $(p \rightarrow q)$, $(\sim p \vee q)$, and $\sim(p \wedge \sim q)$ are equivalent. It is always the case that two compound statements are equivalent if and only if they have the same truth sets. Thus we can test for equivalence by checking whether they have the same Venn diagram.

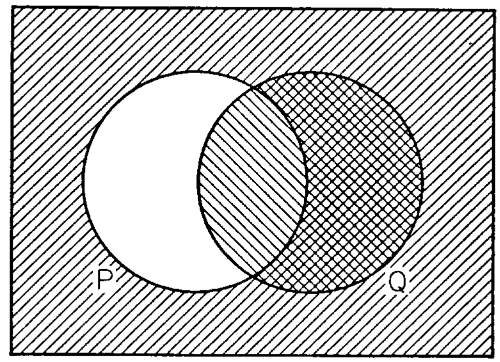


Figure 6

Suppose that p is a statement that is logically true. What is its truth set? Now p is logically true if and only if it is true in every logically possible case, so that the truth set of p must be $\mathcal{U}$. Similarly, if p is logically false, then it is false for every logically possible case, so that its truth set is the empty set $\mathcal{E}$.

Finally, let us consider the implication relation. Recall that p implies q if and only if the conditional $p \rightarrow q$ is logically true. But $p \rightarrow q$ is logically true if and only if its truth set is $\mathcal{U}$, that is, $\widetilde{(P - Q)} = \mathcal{U}$, or $(P - Q) = \mathcal{E}$. From Figure 4 we see that if $P - Q$ is empty, then P is contained in Q. We shall symbolize the containing relation as follows: $P \subset Q$ means "P is a subset of Q." We conclude that $p \to q$ is logically true if and only if $P \subset Q$.

Statement Language	Set Language
r	R
s	S
$\sim r$	$\tilde{R}$
$r \vee s$	$R \cup S$
$r \wedge s$	$R \cap S$
$r \to s$	$\widetilde{(R - S)}$
r implies s	$R \subset S$
r is equivalent to s	$R = S$

Figure 7

Figure 7 supplies a "dictionary" for translating from statement language to set language, and back. To each statement relative to a set of possibilities $\mathcal{U}$ there corresponds a subset of $\mathcal{U}$, namely the truth set of the statement. This is shown in lines 1 and 2 of the figure. To each connective there corresponds an operation on sets, as illustrated in the next four lines. And to each relation between statements there

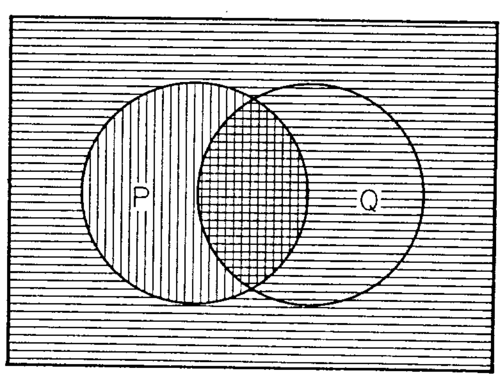
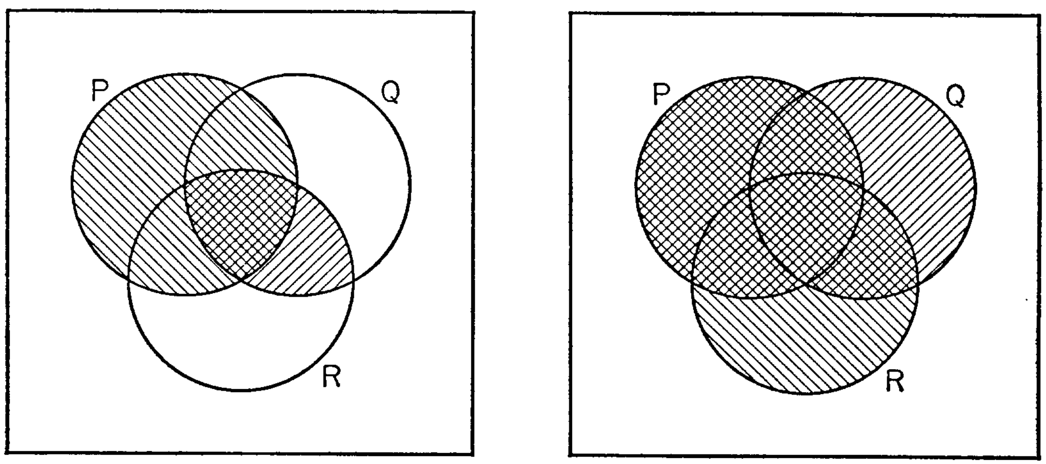


Figure 8

corresponds a relation between sets, examples of which are shown in the last two lines of the figure.

Example 1. Prove by means of a Venn diagram that the statement $[p \vee (\sim p \vee q)]$ is logically true. The assigned set of this statement is $[P \cup (\tilde{P} \cup Q)]$, and its Venn diagram is shown in Figure 8. In that figure the set P is shaded vertically, and the set $\tilde{P} \cup Q$ is shaded horizontally. Their union is the entire shaded area, which is $\mathcal{U}$, so that the compound statement is logically true.

Example 2. Prove by means of Venn diagrams that $p \vee (q \wedge r)$ is equivalent to $(p \vee q) \wedge (p \vee r)$. The truth set of $p \vee (q \wedge r)$ is the entire shaded area of Figure 9a, and the truth set of $(p \vee q) \wedge (p \vee r)$



(a) (b)

Figure 9

is the doubly shaded area in Figure 9b. Since these two sets are equal, we see that the two statements are equivalent.

Example 3. Show by means of a Venn diagram that q implies $p \rightarrow q$. The truth set of $p \rightarrow q$ is the shaded area in Figure 6. Since this shaded area includes the set Q, we see that q implies $p \rightarrow q$.

EXERCISES

Note: In Exercises 1, 2, and 3, find first the truth set of each statement.

1. Use Venn diagrams to test which of the following statements are logically true or logically false.

(a) $p \lor \sim p$.
(b) $p \land \sim p$.
(c) $p \lor (\sim p \land q)$.
(d) $p \to (q \to p)$.
(e) $p \land \sim(q \to p)$.

[*Ans.* (a), (d) logically true; (b), (e) logically false.]

2. Use Venn diagrams to test the following statements for equivalences.
(a) $p \lor \sim q$.
(b) $\sim(p \land q)$.
(c) $\sim(q \land \sim p)$.
(d) $p \to \sim q$.
(e) $\sim p \lor \sim q$.

[*Ans.* (a) and (c) equivalent; (b) and (d) and (e) equivalent.]

3. Use Venn diagrams for the following pairs of statements to test whether one implies the other.
(a) $p; p \land q$.
(b) $p \land \sim q; \sim p \to \sim q$.
(c) $p \to q; q \to p$.
(d) $p \land q; p \land \sim q$.

4. Devise a test for inconsistency of p and q, using Venn diagrams.

5. Three or more statements are said to be inconsistent if they cannot all be true. What does this state about their truth sets?

6. In the following three compound statements (a) assign variables to the components, (b) bring the statements into symbolic form, (c) find the truth sets, and (d) test for consistency.

If this is a good course, then I will work hard in it.
If this is not a good course, then I shall get a bad grade in it.
I will not work hard, but I will get a good grade in this course.

[*Ans.* Inconsistent.]

Note: In Exercises 7–9 assign to each set a statement having it as a truth set.

7. Use truth tables to find which of the following sets are empty.
(a) $(P \cup Q) \cap (\tilde{P} \cup \tilde{Q})$.
(b) $(P \cap Q) \cap (\tilde{Q} \cap R)$.
(c) $(P \cap Q) - P$.
(d) $(P \cup R) \cap (\tilde{P} \cup \tilde{Q})$. [*Ans.* (b) and (c).]

8. Use truth tables to find out whether the following sets are all different.
(a) $P \cap (Q \cup R)$.
(b) $(R - Q) \cup (Q - R)$.
(c) $(R \cup Q) \cap \widetilde{(R \cap Q)}$.

(d) $(P \cap Q) \cup (P \cap R)$.
(e) $(P \cap Q \cap \tilde{R}) \cup (P \cap \tilde{Q} \cap R) \cup (\tilde{P} \cap Q \cap \tilde{R}) \cup (\tilde{P} \cap \tilde{Q} \cap R)$.

9. Use truth tables for the following pairs of sets to test whether one is a subset of the other.

(a) $P; P \cap Q$.
(b) $P \cap \tilde{Q}; Q \cap \tilde{P}$.
(c) $P - Q; Q - P$.
(d) $\tilde{P} \cap \tilde{Q}; P \cup Q$.

10. Show, both by the use of truth tables and by the use of Venn diagrams, that $p \wedge (q \vee r)$ is equivalent to $(p \wedge q) \vee (p \wedge r)$.

11. The *symmetric difference* of P and Q is defined to be $(P - Q) \cup (Q - P)$. What connective corresponds to this set operation?

12. Let p, q, r be a complete set of alternatives (see Chapter I, Section 8). What can we say about the truth sets P, Q, R?

*4. THE ABSTRACT LAWS OF SET OPERATIONS

The set operations which we have introduced obey some very simple abstract laws, which we shall list in this section. These laws can be proved by means of Venn diagrams or they can be translated into statements and checked by means of truth tables.

The abstract laws given below bear a close resemblance to the elementary algebraic laws with which the student is already familiar. The resemblance can be made even more striking by replacing $\cup$ by $+$ and $\cap$ by $\times$. For this reason, a set, its subsets, and the laws of combination of subsets are considered an algebraic system, called a Boolean algebra—after the British mathematician George Boole who was the first person to study them from the algebraic point of view. Any other system obeying these laws, for example, the system of compound statements studied in Chapter I, is also known as a Boolean algebra. We can study any of these systems from either the algebraic or the logical point of view.

Below are the basic laws of Boolean algebras. The proofs of these laws will be left as exercises.

The laws governing union and intersection:

A1. $A \cup A = A$.
A2. $A \cap A = A$.

A3. $A \cup B = B \cup A.$

A4. $A \cap B = B \cap A.$

A5. $A \cup (B \cup C) = (A \cup B) \cup C.$

A6. $A \cap (B \cap C) = (A \cap B) \cap C.$

A7. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$

A8. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$

A9. $A \cup \mathcal{U} = \mathcal{U}.$

A10. $A \cap \mathcal{E} = \mathcal{E}.$

A11. $A \cap \mathcal{U} = A.$

A12. $A \cup \mathcal{E} = A.$

The laws governing complements:

B1. $\tilde{\tilde{A}} = A.$

B2. $A \cup \tilde{A} = \mathcal{U}.$

B3. $A \cap \tilde{A} = \mathcal{E}.$

B4. $\widetilde{(A \cup B)} = \tilde{A} \cap \tilde{B}.$

B5. $\widetilde{(A \cap B)} = \tilde{A} \cup \tilde{B}.$

B6. $\tilde{\mathcal{U}} = \mathcal{E}.$

The laws governing set-differences:

C1. $A - B = A \cap \tilde{B}.$

C2. $\mathcal{U} - A = \tilde{A}.$

C3. $A - \mathcal{U} = \mathcal{E}.$

C4. $A - \mathcal{E} = A.$

C5. $\mathcal{E} - A = \mathcal{E}.$

C6. $A - A = \mathcal{E}.$

C7. $(A - B) - C = A - (B \cup C).$

C8. $A - (B - C) = (A - B) \cup (A \cap C).$

C9. $A \cup (B - C) = (A \cup B) - (C - A).$

C10. $A \cap (B - C) = (A \cap B) - (A \cap C).$

EXERCISES

1. Test laws in the group A1–A12 by means of Venn diagrams.

2. "Translate" the A-laws into laws about compound statements. Test these by truth tables.

3. Test the laws in groups B and C by Venn diagrams.

4. "Translate" the B- and C-laws into laws about compound statements. Test these by means of truth tables.

5. Derive the following results from the 28 basic laws.
   (a) $A = (A \cap B) \cup (A \cap \tilde{B})$.
   (b) $A \cup B = (A \cap B) \cup (A \cap \tilde{B}) \cup (\tilde{A} \cap B)$.
   (c) $A \cap (A \cup B) = A$.
   (d) $A \cup (\tilde{A} \cap B) = A \cup B$.

6. From the A- and B-laws and from C1, derive C2–C6.

7. Use A1–A12 and C2–C10 to derive B1, B2, B3, and B6.

SUPPLEMENTARY EXERCISES

Note: Use the following definitions in these exercises: Let $+$ be symmetric difference (see Section 3, Exercise 11), $\times$ be intersection, let 0 be $\mathcal{E}$ and 1 be $\mathcal{U}$.

8. From A2, A4, and A6 derive the properties of multiplication.

9. Find corresponding properties for addition.

10. Set up addition and multiplication tables for 0 and 1.

11. What do $A \times 0$, $A \times 1$, $A + 0$, and $A + 1$ equal?
[*Ans.* 0; A; A; $\tilde{A}$.]

12. Show that

$$A \times (B + C) = (A \times B) + (A \times C).$$

13. Show that the following equation is not always true.

$$A + (B \times C) = (A + B) \times (A + C).$$

*5. TWO-DIGIT NUMBER SYSTEMS

In the decimal number system one can write any number by using only the ten digits, 0, 1, 2, . . . , 9. Other number systems can be constructed which use either fewer or more digits. Probably the simplest number system is the *binary number system* which uses only the digits 0 and 1. We shall consider all the possible ways of forming number systems using only these two digits.

The two basic arithmetical operations are addition and multiplication. To understand any arithmetic system, it is necessary to know how to add or multiply any two digits together. Thus to understand the decimal system, we had to learn a multiplication table and an addition table, each of which had 100 entries. To understand the binary system, we have to learn a multiplication and an addition table, each of which has only four entries. These are shown in Figure 10.

+	0	1
0	0	1
1	1	?

•	0	1
0	0	0
1	0	1

Figure 10

The multiplication table given there is completely determined by the two familiar rules that multiplying a number by zero gives zero, and multiplying a number by one leaves it unchanged. For addition, we have only the rule that the addition of zero to a number does not change that number. The latter rule is sufficient to determine all but one of the entries in the addition table in Figure 10. We must still decide what shall be the sum $1 + 1$.

What are the possible ways in which we can complete the addition table? The only one-digit numbers that we can use are 0 and 1, and these lead to interesting systems. Of the possible two-digit numbers, we see that 00 and 01 are the same as 0 and 1 and so do not give anything new. The number 11 or any greater number would introduce a "jump" in the table, hence the only other possibility is 10. The addition tables of these three different number systems are shown in Figure 11, and they all have the multiplication table shown in Figure 10. Each of these systems is interesting in itself as the interpretations below show.

Let us say that the *parity* of a positive integer is the fact of it being odd or even. Consider now the number system having the addition table in Figure 11a and let 0 represent "even" and 1 represent "odd."

+	0	1
0	0	1
1	1	0

(a)

+	0	1
0	0	1
1	1	1

(b)

+	0	1
0	0	1
1	1	10

(c)

Figure 11

The tables above now tell how the parity of a combination of two positive integers is related to the parity of each. Thus $0 \cdot 1 = 0$ tells us that the product of an even number and an odd number is even, while $1 + 1 = 0$ tells us that the sum of two odd numbers is even, etc. Thus the first number system is that which we get from the arithmetic of the positive integers if we consider only the parity of numbers.

The second number system, which has the addition table in Figure 11b, has an interpretation in terms of sets. Let 0 correspond to the empty set $\mathcal{E}$ and 1 correspond to the universal set $\mathcal{U}$. Let the addition of numbers correspond to the union of sets and let the multiplication of sets correspond to the intersection of sets. Then $0 \cdot 1 = 0$ tells us that $\mathcal{E} \cap \mathcal{U} = \mathcal{E}$ and $1 + 1 = 1$ tells us that $\mathcal{U} \cup \mathcal{U} = \mathcal{U}$. The student should give the interpretations for the other arithmetic computations possible for this number system.

Finally, the third number system, which has the addition table in Figure 11c, is the so-called *binary number system.* Every ordinary integer can be written as a binary integer. Thus the binary 0 corresponds to the ordinary 0, and the binary unit 1 to the ordinary single unit. The binary number 10 means a "unit of higher order" and corresponds to the ordinary number two (not to ten). The binary number 100 then means two times two or four. In general, if $b_n b_{n-1} \ldots b_2 b_1 b_0$ is a binary number, where each digit is either 0 or 1, then the corresponding ordinary integer I is given by the formula

$$I = b_n \cdot 2^n + b_{n-1} \cdot 2^{n-1} + \ldots + b_2 \cdot 2^2 + b_1 \cdot 2 + b_0.$$

Thus the binary number 11001 corresponds to $2^4 + 2^3 + 1 = 16 + 8 + 1 = 25$. The table in Figure 12 shows some binary numbers and their decimal equivalents.

Binary number	1	10	11	100	101	110	111	1000	10000	100000
Decimal equiv.	1	2	3	4	5	6	7	8	16	32

Figure 12

Because electronic circuits are particularly well adapted to performing computations in the binary system, modern high-speed electronic computers are frequently constructed to work in the binary system.

Example. As an example of a computation, let us multiply 5 by 5 in the binary system. Since the binary equivalent of 5 is the number 101, the multiplication is done as follows.

```
  101
  101
  ---
  101
 000
101
-----
11001
```

The answer is the binary number 11001, which we saw above was equivalent to the decimal integer 25, the answer we expected to get.

EXERCISES

1. Complete the interpretations of the addition and multiplication tables for the number systems representing (a) parity, (b) the sets $\mathcal{U}$ and $\mathcal{E}$.

2. (a) What are the binary numbers corresponding to the integers 11, 52, 64, 98, 128, 144? [*Partial Ans.* 1100010 corresponds to 98.]

(b) What decimal integers correspond to the binary numbers 1111, 1010101, 1000000, 11011011?

[*Partial Ans.* 1010101 corresponds to 85.]

3. Carry out the following operations in the binary system. Check your answer.

(a) $29 + 20$.

(b) $9 \cdot 7$.

4. Of the laws listed below, which apply to each of the three systems?

(a) $x + y = y + x$.

(b) $x + x = x$.

(c) $x + x + x = x$.

5. Interpret $a + b$ to be the larger of the two numbers a and b, and $a \cdot b$ to be the smaller of the two. Write tables of "addition" and "multiplication" for the digits 0 and 1. Compare the result with the three systems given above.

[*Ans.* Same as the $\mathcal{U}$, $\mathcal{E}$ system.]

6. What do the laws A1–A10 of the last section tell us about the second number system established above?

7. The first number system above (about parity) can be interpreted to deal with the remainders of integers when divided by 2. An even number leaves 0, an odd number leaves 1. Construct tables of addition and multiplication for remainders of integers when divided by 3. [*Hint:* These will be 3×3 tables.]

8. Given a set of four elements, suppose that we want to number its subsets. For a given subset, write down a binary number as follows: The first digit is 1 if and only if the first element is in the subset, the second digit is 1 if and only if the second element is in the subset, etc. Prove that this assigns a unique number, from 0 to 15, to each subset.

9. In a multiple choice test the answers were numbered 1, 2, 4, and 8. The students were told that there might be no correct answer, or that one or more answers might be correct. They were told to *add* together the numbers of the correct answers (or to write 0 if no answer was correct).

(a) By using the result of Exercise 8, show that the resulting number gives the instructor all the information he wants.

(b) On a given question the correct sum was 7. Three students put down 4, 8, and 15, respectively. Which answer was most nearly correct? Which answer was worst? [*Ans.* 15 best, 8 worst.]

10. In the ternary number system, numbers are expressed to the base 3, so that 201 in this system stands for $2 \cdot 3^2 + 0 \cdot 3 + 1 \cdot 1 = 19$.

(a) Write the numbers from 1 through 30 in this notation.

(b) Construct a table of addition and multiplication for the digits 0, 1, 2.

(c) Carry out the multiplication of $5 \cdot 5$ in this system. Check your answer.

11. Explain the meaning of the numeral "2907" in our ordinary (base 10) notation, in analogy to the formula *I* given for the binary system.

12. Show that the addition and multiplication tables set up in Section 4, Exercise 10 correspond to one of our three systems.

*6. VOTING COALITIONS

As an application of our set concepts, we shall consider the significance of voting coalitions in voting bodies. Here the universal set is a set of human beings which form a decision-making body. For example, the universal set might be the members of a committee, or of a city council, or of a convention, or of the House of Representatives, etc. Each member can cast a certain number of votes. The decision as to whether or not a measure is passed can be decided by a simple majority rule, or $\frac{2}{3}$ majority, etc.

Suppose now that a subset of the members of the body forms a coalition in order to pass a measure. The question is whether or not they have enough votes to guarantee passage of the measure. If they have enough votes to carry the measure, then we say they form a *winning coalition*. If the members *not* in the coalition can pass a measure of their own, then we say that the original coalition is a *losing coalition*. Finally, if the members of the coalition cannot carry their measure, and if the members not in the coalition cannot carry their measure, then the coalition is called a *blocking coalition*.

Let us restate these definitions in set-theoretic terms. A coalition C is winning if they have enough votes to carry an issue; coalition C is losing if the coalition $\tilde{C}$ is winning; and coalition C is blocking if neither C nor $\tilde{C}$ is a winning coalition.

The following facts are immediate consequences of these definitions. The complement of a winning coalition is a losing coalition. The complement of a losing coalition is a winning coalition. The complement of a blocking coalition is a blocking coalition.

Example 1. A committee consists of six men each having one vote. A simple majority vote will carry an issue. Then any coalition of four or more members is winning, any coalition with one or two members is losing, and any three-person coalition is blocking.

Example 2. Suppose in Example 1 one of the six members (say the chairman) is given the additional power to break ties. Then any three-person coalition of which he is a member is winning, while the other three-person coalitions are losing; hence there are no blocking coalitions. The other coalitions are as in Example 1.

Example 3. Let the universal set $\mathcal{U}$ be the set $\{x, y, w, z\}$, where x and y each has one vote, w has two votes, and z has three votes. Suppose it takes five votes to carry a measure. Then the winning coalitions are: $\{z, w\}$, $\{z, x, y\}$, $\{z, w, x\}$, $\{z, w, y\}$, and $\mathcal{U}$. The losing coalitions are the complements of these sets. Blocking coalitions are: $\{z\}$, $\{z, x\}$, $\{z, y\}$, $\{w, x\}$, $\{w, y\}$, and $\{w, x, y\}$.

The last example shows that it is not always necessary to list all members of a winning coalition. For example, if the coalition $\{z, w\}$ is winning, then it is obvious that the coalition $\{z, w, y\}$ is also winning. In general, if a coalition C is winning, then any other set that has C as a subset will also be winning. Thus we are led to the notion of a *minimal winning coalition*. A minimal winning coalition is a winning coalition which contains no smaller winning coalition as a subset. Another way of stating this is that a minimal winning coalition is a winning coalition such that, if any member is lost from the coalition, then it ceases to be a winning coalition.

If we know the minimal winning coalitions, then we know everything that we need to know about the voting problem. The winning coalitions are all those sets that contain a minimal winning coalition, and the losing coalitions are the complements of the winning coalitions. All other sets are blocking coalitions.

In Example 1 the minimal winning coalitions are the sets containing four members. In Example 2 the minimal winning coalitions are the three-member coalitions that contain the tie-breaking member and the four-member coalitions that do not contain the tie-breaking member. The minimal winning coalitions in the third example are the sets {z, w} and {z, x, y}.

Sometimes there are committee members who have special powers or lack of power. If a member can pass any measure he wishes without needing anyone else to vote with him, then we call him a *dictator*. Thus member x is a dictator if and only if {x} is a winning coalition. A somewhat weaker but still very powerful member is one who can by himself block any measure. If x is such a member, then we say that x has *veto power*. Thus x has veto power if and only if {x} is a blocking coalition. Finally if x is not a member of any minimal winning coalition, we shall call him a *powerless* member. Thus x is powerless if and only if any winning coalition of which x is a member is a winning coalition without him.

Example 4. An interesting example of a decision-making body is the Security Council of the United Nations. The Security Council has eleven members consisting of the five permanent large-nation members called the Big Five, and six small-nation members. In order that a measure be passed by the Council, seven members including all of the Big Five must vote for the measure. Thus the seven-member sets made up of the Big Five plus two small nations are the minimal winning coalitions. Then the losing coalitions are the sets that contain at most four small nations. The blocking coalitions are the sets that are neither winning nor losing. In particular, a unit set that contains one of the Big Five as a member is a blocking coalition. This is the sense in which a Big Five member has a veto. [The possibility of "abstaining" is immaterial in the above discussion.]

In 1966 the number of small-nation members was increased to 10. A measure now requires the vote of nine members, including all of the Big Five. (See Exercise 11.)

EXERCISES

1. A committee has w, x, y, and z as members. Member w has two votes, the others have one vote each. List the winning, losing, and blocking coalitions.

2. A committee has *n* members, each with one vote. It takes a majority vote to carry an issue. What are the winning, losing, and blocking coalitions?

3. The Board of Estimate of New York City consists of eight members with voting strength as follows:

s.	Mayor	4 votes
t.	Controller	4
u.	Council President	4
v.	Brooklyn Borough President	2
w.	Manhattan Borough President	2
x.	Bronx Borough President	2
y.	Richmond Borough President	2
z.	Queens Borough President	2

A simple majority is needed to carry an issue. List the minimal winning coalitions. List the blocking coalitions. Do the same if we give the mayor the additional power to break ties.

4. A company has issued 100,000 shares of common stock and each share has one vote. How many shares must a stockholder have to be a dictator? How many to have a veto? [*Ans.* 50,001; 50,000.]

5. In Exercise 4, if the company requires a $\frac{2}{3}$ majority vote to carry an issue, how many shares must a stockholder have to be a dictator or to have a veto? [*Ans.* At least 66,667; at least 33,334.]

6. Prove that if a committee has a dictator as a member, then the remaining members are powerless.

7. We can define a maximal losing coalition in analogy to the minimal winning coalitions. What is the relation between the maximal losing and minimal winning coalitions? Do the maximal losing coalitions provide all relevant information?

8. Prove that any two minimal winning coalitions have at least one member in common.

9. Find all the blocking coalitions in the Security Council example.

10. Prove that if a man has veto power and if he together with any one other member can carry a measure, then the distribution of the remaining votes is irrelevant.

11. Find the winning, losing, and blocking coalitions in the Security Council, using the revised (1966) structure.

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