A Clap Can Chirp: Waves and Echoes in the Racquetball Court

Vissuta Jiwariyavej *Women In Science Project*, Alexander Barnett *Advisor, Mathematics Department*, *Dartmouth College*

Introduction

If you stand inside a racquetball court and clap your hands, what you will hear is not a simple echo of the clap. A handclap will generate a chirped echo, which is a rising pitch.

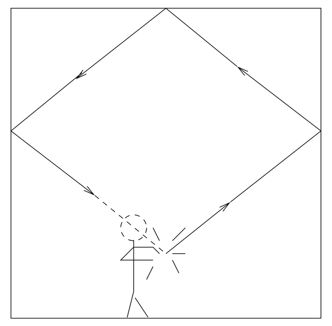

Figure 1: A person clapping in the raquetball court. The wave signal travels, reflects off the wall and finally comes back to the person.

Such an echo is generated by superposition of the reflections of the clap off the walls. In order for the sound wave from the clap to travel back to the listener, the wave needs to reflect off the walls and bounce back to the source. However, since the sound wave travels in all direction from the source, there exists more than one reflection path. When all reflections are combined, an irregular echo is created, which produces a chirp.

In this project, we are studying the chirping effect using a mathematical approach to simulate the signal and compare to the recordings from the actual clap in the racquetball court.

Similar strange chirp echoes have recently been reported. . .
"Now I have heard echoes in my life, but this was really strange," says David Lubman, an acoustical engineer, after hearing the echo of his handclap in front of Maya Pyramid.

Simple 2-D Model

First, we consider the cross-sectional area of the racquetball court which is in the square shape, size 20'x20'. Let a person stand in the middle of the room clapping. The sound wave travels from the source in all directions and reflects off the wall. The reflections which finally come back to the source can be found by considering a lattice of image sources (see Figure). Each image source contributes some certain amount of energy to the total sound wave. However, because the image sources are at different distance from the real source, the amount of time the wave taken to travel back to the person are also different. For the ideal situation, the image sources of the same distance away from the source contributes the same amount of energy and at the same time. Therefore, finding how many sources contribute to the signal (in a 2-D simulation) is equivalent to counting the number of solutions (x, y) to the equation $n = x^2 + y^2$ for a given n, where x, y are integers.

Definition 1 *$r_2(n)$ is the number of representations of n by 2 squares, allowing zeros and distinguishing signs and order. In other words,* $r_2(n) = \#\{(x, y) : x^2 + y^2 = n\}$

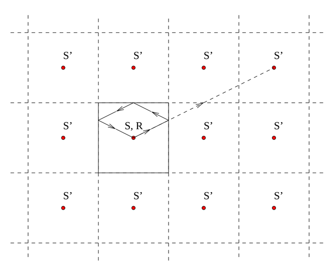

Figure 2: Actual source (S), lattice of image sources (S'), and receiver (R)

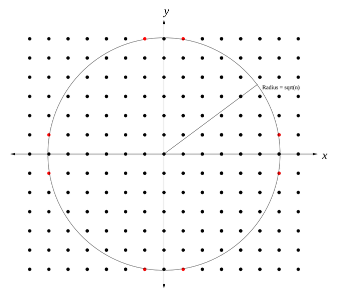

Figure 3: Sum of squares Given a circle whose radius is $\sqrt{n}$, the coordinates (x, y) of all points on the circle are solutions of the equation $n = x^2 + y^2$

We use the formulae for $r_2(n)$ in the next Section to count the contribution of each reflection, and then generate the sound wave. Assuming a clap is a simple pulse (delta function $\delta(t)$), then one reflection is also a simple pulse. A reflection pulse comes back to the source at different times, and by accumulating such pulses we produce a chirp sound similar to what we hear in the court.

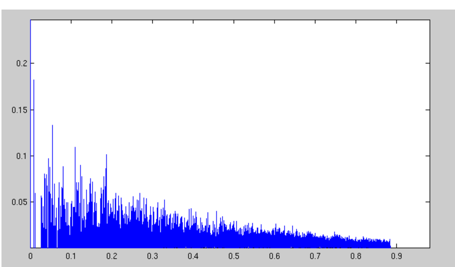

Figure 4: Simulated sound signal of a clap

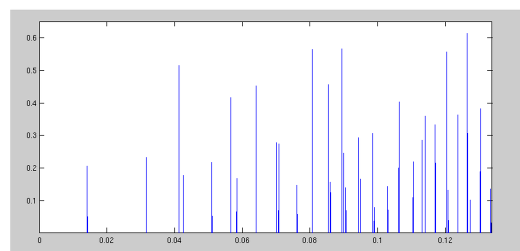

Figure 5: Part of the simulated sound wave. Each spike represents the amount of energy which reaches the listener at time t. The number of source contributing energy at the same time is counted by using formulae for $r_2(n)$

Then we use fourier series to compute the spectogram of the wave we generated.

Formulae for the Number of Sums of Square

Another way to count the number of image sources which are at the same distance are to use the following formulae.

Definition 2 *Let $n = 2^{a_0} p_1^{2a_1} \dots p_m^{2a_m} q_1^{b_1} \dots q_r^{b_r}$ Where the p_is are primes of the form 4k+3 and the q_is are primes of the form 4k+1.*

Proposition 1 *[1]*

$$\textit{Define } B = (b_1 + 1) \dots (b_r + 1) \tag{1}$$

The number of representations of n as the sum of two squares is given by

$$r_2(n) = \begin{cases} 0 & \text{if any } a_i \text{ is a half-integer} \\ 4B & \text{if all } a_i \text{ are integers} \end{cases} \tag{2}$$

Definition 3 *Dirichlet characters modulo m, χ is defined by*

$$\chi(d) = \begin{cases} 1 & \text{if } d \equiv 1 \pmod 4 \\ -1 & \text{if } d \equiv 3 \pmod 4 \\ 0 & \text{otherwise} \end{cases}$$

Proposition 2 *[2] The number of integral solutions (x, y) where $x > 0, y \geq 0$ to the equation $x^2+y^2 = n$ is $\sum_{d|n} \chi(d)$, where the sum is over all divisors of n.*

Since $x > 0$ and $y \geq 0$, the proposition only represents a quarter of the xy-plane as shown in the figure. By rotating this region by the angle of $\pi/2$, the entire xy-plane is covered. The number of solutions of $x^2 + y^2 = n$ is now $4\sum_{d|n} \chi(d)$ or

$$r_2(n) = 4 \sum_{d|n} \chi(d)$$

Proof of Equivalence

Our goal is to prove that proposition 1 and 2 are equivalent. From n in definition 2, then an arbitrary factor of n can be written $d = 2^{\alpha_0} p_1^{\alpha_1} \dots p_m^{\alpha_m} q_1^{\beta_1} \dots q_r^{\beta_r}$, where α 's and β 's are integers, which $0 \leq \alpha_0 \leq 2a_0, 0 \leq \alpha_1 \leq 2a_1, \dots, 0 \leq \alpha_m \leq 2a_m, 0 \leq \beta_1 \leq b_1, \dots, 0 \leq \beta_r \leq b_r$.

By the fundamental theorem of arithmetic, stating that every natural number greater than 1 can be written as a unique product of prime numbers, each set of $\{\alpha_0, \alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_r\}$ results in a distinct factor. Since each member of the set can be chosen independently, therefore the number of ways to construct a factor of n is

$$(a_0 + 1)(2a_1 + 1) \dots (2a_m + 1)(b_1 + 1) \dots (b_r + 1)$$

For d that can be written in the form of $d = 2k$, 2 needs to be a factor of d. So $1 \leq \alpha_0 \leq a_0$. Therefore, the number of factors d in the form of $d = 2k$ is

$$a_0(2a_1 + 1) \dots (2a_m + 1)(b_1 + 1) \dots (b_r + 1)$$

According to the definition of χ (d) in Proposition 2, χ of even number is zero. Therefore, even factors don't contribute to the sum in proposition 2. So from now on we will consider χ value of odd factors only. The rest of the factors are odd factors, which therefore number

$$(2a_1 + 1) \dots (2a_m + 1)(b_1 + 1) \dots (b_r + 1)$$

Now, we try to catagorize the odd factors in two groups, ones that can be written in the form of $4k + 1$, and ones that can be written in the form of $4k + 3$. From

$$\begin{aligned} (4k_1 + 1)(4k_2 + 1) &\equiv 1 \cdot 1 \equiv 1 \pmod 4 \\ (4k_1 + 1)(4k_2 + 3) &\equiv 1 \cdot 3 \equiv 3 \pmod 4 \\ (4k_1 + 3)(4k_2 + 3) &\equiv 3 \cdot 3 \equiv 1 \pmod 4 \end{aligned}$$

then $q_1^{\beta_1} \dots q_r^{\beta_r} \equiv 1 \pmod 4$ for any $\beta_1, \dots, \beta_r$. So it is only the powers of $p_1, \dots, p_m$ which determines whether $d \equiv 1$ or $3 \pmod 4$

We first show $\sum_{d|n} \chi(d) = 0$ if any of the powers $2a_1, 2a_1, \dots, 2a_m$ appearing in definition 2 are odd.

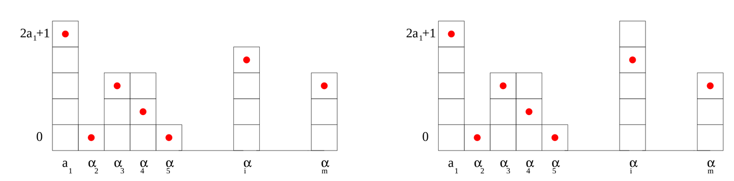

Figure 6: Combinations of power of prime factors in the form $4k + 1$ (**left**) when there is at least one prime which has even power and (**right**) when all prime has odd power.

Let $2a_i$ be odd, in other word, a_i is a half-integer. Then the number of choice of α_i is even. For each choice of other α's, the parity of the sum of power of p's when $\alpha_i = 0$ and when $\alpha_i = 1$ are opposite. So among when $\alpha_i = 0$ and $\alpha_i = 1$, the number of factors of n with odd and even sum of power are equals. And so the factors of n when $\alpha_i = 2$ and $\alpha_i = 3$ can be paired up with opposite parity, and in the same way, the factors of n when $\alpha_i = 2a_i - 1$ and $\alpha_i = 2a_i$ can also be paired up with opposite parity.

Since $\sum_{d|n} = \#\{d : d = 4k + 1\} - \#\{d : d = 4k + 3\}$ and the number of both catagories of prime numbers are equal, therefore,$\sum_{d|n} \chi(d) = 0$.

The only situation left to consider is the case of when the power of prime p_i's are all even. In this case we will show that $\sum_{d|n} \chi(d) = B$

α_1 can be $0, 1, 2, \dots, 2a_1$. Similar to the previous case, $\alpha_1 = 1$ can be paired up with $\alpha_1 = 2$ to cancel the value of the sum of $\chi(d)$, and the same pattern continues to $\alpha_1 = 2a_1 - 1$ and $\alpha_1 = 2a_1$. So the only case left to consider is when $\alpha_1 = 0$.

When $\alpha_1 = 0$ $\alpha_2 = 1$ can be paired up with $\alpha_2 = 2$, and the same pattern continues to $\alpha_2 = 2a_2 - 1$ and $\alpha_2 = 2a_2$. So the only case left to consider is when $\alpha_2 = 0$.

Continuing in this fasion, all the non-cancelling terms in the sum are accounted for by the case $\alpha_1 = \alpha_2 = \dots = \alpha_m = 0$

When all α_i's is 0, the factor of n that can be constructed is in the form $d = 2^0 p_1^0 \dots p_m^0 q_1^{b_1} \dots q_r^{b_r}$ which equals $q_1^{b_1} \dots q_r^{b_r}$. The number of such choices of $\beta_1, \dots, \beta_r$ is $(b_1 + 1) \dots (b_r + 1) = B$. QED.

Spectrogram Comparison and 3-D model

A *spectrogram* is a plot which shows the frequency content of a signal as it changes in time. The axes are time and frequency. It exctracts similar information to the human ear. Therefore we can study the behavior of the sound waves by comparing the spectrograms.

In order to see how well the simulation can model the phenomenon, we also use fourier series and spectrogram to analyze the actual sound recordings. Then we compare the spectograms of the simulated sound waves and the actual recordings.

In the spectrogram we see certain straight lines rising with various slopes. The lines occur because of the frequency spacing, giving the slope of the line equals to multiples of $\frac{2c^2}{L^2}$, where c is the velocity of sound and L is the distance between image sources.

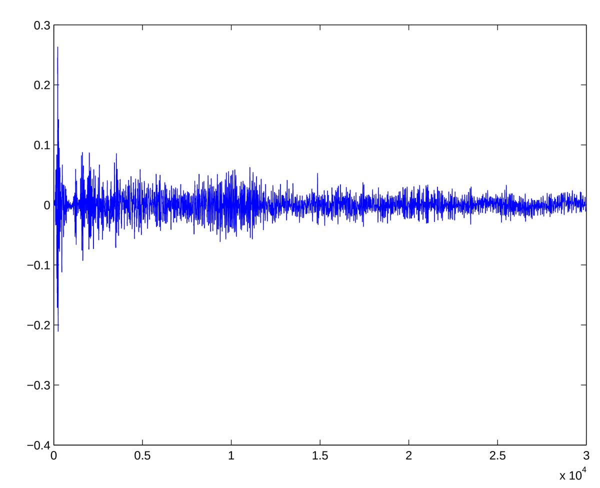

Figure 7: Sound signal of the actual recording of a clap

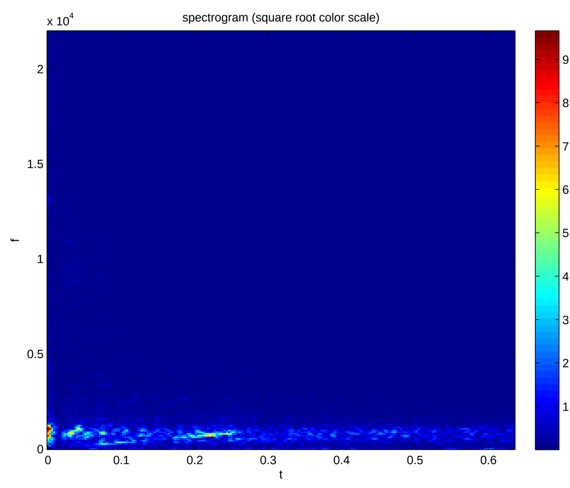

Figure 8: Spectrogram analysis of the actual recording of a clap generated

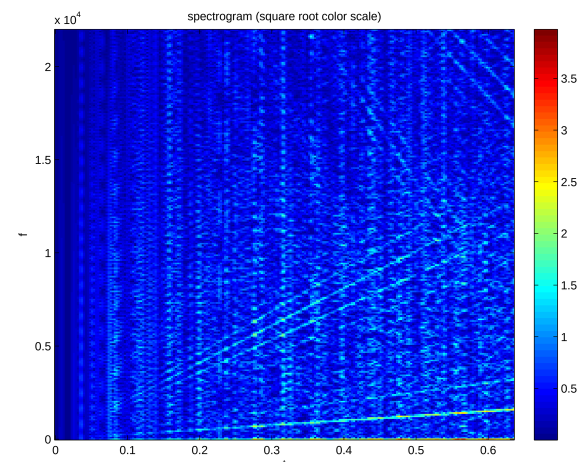

Figure 9: Spectrogram generated from the signal of 2-D model

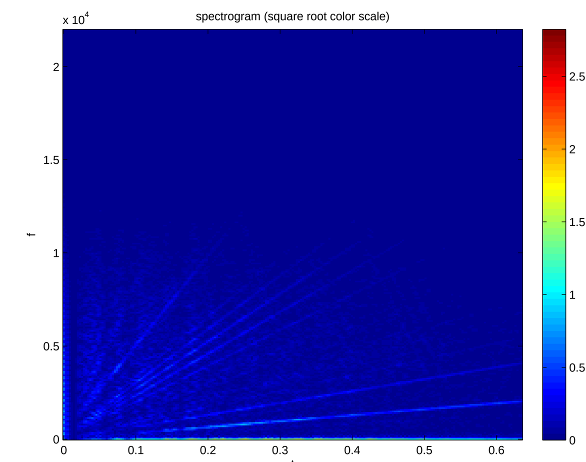

Figure 10: Spectrogram generated from the signal of 3-D model

The 2-D simulation seems to still allow improvement of the result, therefore we move on to the 3-D simulation using the same method as 2-D simulation. In 3-D simulation, we also experiment with the source and the receiver whose positions are not exactly at the middle of the racquetball court. The slope of the lines from 3-D simulation seems to match the lines from the actual recordings better than the 2-D simulation as expected.

Future Direction

The 3-D model which we found to quite well represent the real phenomenon ensures that our study is heading towards the right direction. However, there are also unexpected behavior in the actual recording. For example, the strength of the signal seems to be strongly dependent on position of sources and receivers: at some position, the recording of the chirp is loud and clear whereas we could barely hear the chirp at some other position. Also, in contrast to previous results, the spectrogram of some recordings, the chirps create absent lines instead of bright lines. Such aspects of the result might be potential direction to continue the research.

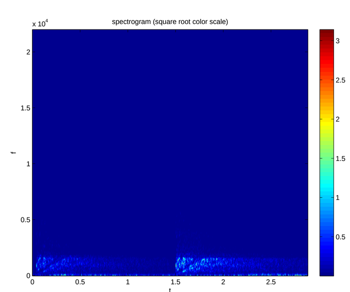

Figure 11: Spectrogram of a recorded signal, showing absent lines instead of bright lines

References

[1] Beiler, A. H. *Recreations in the Theory of Numbers: The Queen of Mathematics Entertains.* New York: Dover, 1966.

[2] Ireland, Kenneth F. *A classical introduction to modern number theory.* New York: Springer-Verlag, 1990.